

TRIANGULATIONS OF ROOT POLYTOPES

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ABSTRACT. Let Φ be an irreducible crystallographic root system and \mathcal{P} its root polytope, i.e., its convex hull. We provide a uniform construction, for all root types, of a triangulation of the facets of \mathcal{P} . We also prove that, on each orbit of facets under the action of the Weyl group, the triangulation is unimodular with respect to a root sublattice that depends on the orbit.

1. INTRODUCTION

Let Φ be an irreducible crystallographic root system in a Euclidean space E , Φ^+ a positive system of Φ , and W the Weyl group of Φ . We denote by \mathcal{P} the *root polytope* associated with Φ , i.e. the convex hull of all roots in Φ .

In [5] we study a natural set of representatives of the faces of \mathcal{P} modulo the action of W , that we call the standard parabolic faces of \mathcal{P} . The set of all roots contained in a standard parabolic face is an abelian ideal of Φ^+ (see Subsection 2.3 for a definition). We call *face ideals* or *facet ideals* the abelian ideals of Φ^+ corresponding to the standard parabolic faces or facets of \mathcal{P} .

In [6], for Φ of type A_n and C_n , we have constructed a triangulation of the standard parabolic facets whose simplexes have a natural interpretation in terms of the corresponding facet ideals. The construction is formally equal for both root types, though the proofs are distinct and based on the special combinatorics of these two root systems and their maximal abelian ideals. Clearly, through the action of W , a triangulation of all the standard parabolic facets can be extended to a triangulation of the boundary of \mathcal{P} . Such an extension is not unique and corresponds to a choice of representatives of the left cosets of W modulo the stabilizers of the standard parabolic facets. The triangulations of the boundary of \mathcal{P} are also studied in [17, 18], for C_n , and in [1] for all classical root types, using the coordinate description of Φ . In [11], the triangulations of the positive root polytope \mathcal{P}^+ , i.e the convex hull of the positive roots and the origin, are studied for Φ of type A_n .

In this paper, we give a uniform construction of a triangulation of the standard parabolic facets, for all finite irreducible crystallographic root system. The construction coincides with the one of [6] for the types A_n and C_n . We also obtain unimodularity results similar to those obtained for A_n and C_n .

We need some preliminaries for describing the results in more detail. If $\beta_1, \beta_2, \gamma_1, \gamma_2 \in \Phi^+$ are such that $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$, we say that $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ are *crossing pairs*. We first prove that if $\{\beta_1, \beta_2\}, \{\gamma_1, \gamma_2\}$ are crossing pairs contained in a (common) abelian ideal, then, for all i, j in $\{1, 2\}$, the differences $\beta_i - \gamma_j$ are roots, in particular β_i and γ_j are comparable. This implies that the set $\{\beta_1, \beta_2, \gamma_1, \gamma_2\}$ has a minimum and a maximum, more precisely, one of the two crossing pairs consists of these minimum and maximum, i.e., either $\beta_1 < \gamma_i < \beta_2$ for both $i = 1$ and 2 , or the analogous relation with β and γ interchanged holds. We define the relations \lesssim and \sim on Φ^+ as follows. For all β_1, β_2 in Φ^+ , we write $\beta_1 \lesssim \beta_2$ if there exist γ_1, γ_2 such that $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$ and $\beta_1 < \gamma_i < \beta_2$ for both $i = 1$ and 2 . Moreover, we write $\beta_1 \sim \beta_2$ if $\beta_1 \lesssim \beta_2$ or $\beta_2 \lesssim \beta_1$. Finally, we say that a subset R of Φ^+ is *reduced* if $\beta_1 \not\sim \beta_2$ for all $\beta_1, \beta_2 \in R$.

The first of main results in this paper is that the maximal reduced subsets in a facet ideal provide a triangulation of the corresponding standard parabolic facet. For each standard parabolic facet F of \mathcal{P} , let I_F be the corresponding facet ideal:

$$I_F = F \cap \Phi,$$

and

$$\mathcal{T}_F = \{\text{Conv}(R) \mid R \subseteq I_F, R \text{ maximal reduced}\},$$

where $\text{Conv}(R)$ is convex hull of R . Then the following result holds.

Theorem 1.1. *For each standard parabolic facet F of \mathcal{P} , \mathcal{T}_F is a triangulation of F .*

Clearly, the set of vertexes of the above triangulation is the set of all roots contained in F .

Theorem 1.1 says, in particular, that the maximal reduced subsets in I_F are linear bases of E . Let Π and θ be the simple system and the highest root of Φ^+ . Then, $\{-\theta\} \cup \Pi$ is the set of vertexes of the affine Dynkin diagram of Φ . For each $\alpha \in \Pi$, let Φ_α and $\tilde{\Phi}_\alpha$ be the root subsystems of Φ generated by $\Pi \setminus \{\alpha\}$ and $\{-\theta\} \cup (\Pi \setminus \{\alpha\})$, respectively, and Φ_α^+ and $\tilde{\Phi}_\alpha^+$ their positive systems contained in Φ^+ . Clearly, $\tilde{\Phi}_\alpha$ has the same rank as Φ . We call the $\tilde{\Phi}_\alpha$, for all $\alpha \in \Pi$, the *standard equal rank* subsystems of Φ . It is known that the standard parabolic facets of \mathcal{P} correspond to the *irreducible* standard equal rank root subsystems of Φ . In fact, for each $\alpha \in \Pi$ such that $\tilde{\Phi}_\alpha$ is irreducible, let

$$I_\alpha = \tilde{\Phi}_\alpha^+ \setminus \Phi_\alpha.$$

Then I_α is a facet ideal of Φ^+ , and each facet ideal of Φ^+ is obtained in this way (see [5]). We prove the following result.

Theorem 1.2. *Let $\alpha \in \Pi$ be such that $\tilde{\Phi}_\alpha$ is irreducible. Then, each maximal reduced subset contained in the facet ideal I_α is a \mathbb{Z} -basis of the root lattice of $\tilde{\Phi}_\alpha$. In particular, all the simplexes of the triangulation \mathcal{T}_F have the same volume.*

Part of the proofs require a case by case analysis. The cases to be considered can be restricted to a special, proper subset of facet ideals. Indeed, the results of [5] imply that the facet ideal I_α ($\alpha \in \Pi$, $\tilde{\Phi}_\alpha$ irreducible), is an *abelian nilradical* (see Subsection 2.4) in the root subsystem $\tilde{\Phi}_\alpha^+$, hence we may reduce to the case of abelian nilradicals.

The case by case analysis is contained in the proof of Proposition 5.12. This proof also provides an algorithm for the explicit computation of the triangulations for each root type, which will be done in a next paper.

2. PRELIMINARIES

In this section we fix our main notation and recall some preliminary results. For the basic preliminary notions, we refer to [2] and [13] for root systems, and to [3] and [12] for Lie algebras.

2.1. Basic notation. General. We sometimes use the symbol $:=$ for emphasizing that equality holds by definition or that we are defining the left term of equality. We denote by $(\ , \)$ the scalar product of E and by $|\ |$ the corresponding norm. We identify E with its dual space, through $(\ , \)$. The null vector of E is denoted by $\underline{0}$. For any $S \subseteq E$, $\text{Span}(S)$ is the vector subspace generated by S over \mathbb{R} (the field of real numbers), and $\text{rk}(S) := \dim \text{Span}(S)$.

Root systems. The simple system of Φ corresponding to the positive system Φ^+ is denoted by Π , while Ω^\vee is the set of fundamental co-weights of Φ , i.e., the dual basis of Π in E . For each $\alpha \in \Pi$, $\tilde{\omega}_\alpha$ is the fundamental co-weight defined by the conditions $(\alpha, \tilde{\omega}_\alpha) = 1$ and $(\alpha', \tilde{\omega}_\alpha) = 0$ for all $\alpha' \in \Pi \setminus \{\alpha\}$. For all $\alpha \in \Pi$ and $\beta \in \Phi$, $c_\alpha(\beta)$ is the coefficient of α in β , i.e.,

$$c_\alpha(\beta) = (\beta, \tilde{\omega}_\alpha).$$

The highest root in Φ^+ is denoted by θ and its coefficients with respect to Π by m_α , thus

$$\theta = \sum_{\alpha \in \Pi} m_\alpha \alpha.$$

We will call m_α the *multiplicity of α in Φ^+* .

For all $\beta \in \Phi$, β^\vee is the corresponding coroot, i.e., $\beta^\vee = \frac{2\beta}{(\beta, \beta)}$.

For each root subsystem Ψ of Φ we set $\Psi^+ = \Psi \cap \Phi^+$. It is well known that Ψ^+ is a positive system for Ψ : we call it the *standard positive system* of Ψ . Moreover, we denote by $L(\Psi)$ and $L^+(\Psi)$ the root lattice and positive root lattice of Ψ , i.e. the \mathbb{Z} -span of Ψ and the \mathbb{N} -span of Ψ^+ , respectively, where \mathbb{Z} and \mathbb{N} are the sets of integers and non-negative integers.

For any $S \subseteq \Phi$, we denote by $\Phi(S)$ the root subsystem of Φ generated by S , i.e., the minimal root system containing S , and we write $\Phi^+(S)$ for $\Phi(S)^+$.

Posets. As usual, \leq denotes both the order of \mathbb{R} and the partial order of E associated to Φ^+ : for all $x, y \in E$, $x \leq y$ if and only if $y - x \in L^+(\Phi)$. We call this last order *the*

standard partial order. We will need only the restriction of the standard partial order to Φ^+ . For any $S \subseteq \Phi^+$, we denote by $\text{Min } S$ and $\text{Max } S$, with capital M, the sets of minimal and maximal elements of S , and by $\min S$ and $\max S$ its possible minimum and maximum, with respect to \leq . The analogous objects with respect to any other order relation \preceq , will be distinguished by the subscript \preceq . The elements in $\text{Min } S \cup \text{Max } S$ are called *the extremal elements of P* . We say that S is *saturated* if it is saturated with respect to the standard partial order, i.e., for all $\beta_1, \beta_2 \in S$ such that $\beta_1 \leq \beta_2$, all the interval $[\beta_1, \beta_2] := \{\gamma \in \Phi \mid \beta_1 \leq \gamma \leq \beta_2\}$ is contained in S . Any subset S' of S is called *an initial section of S* if for all $\beta \in S'$ and $\gamma \in S$, if $\gamma \preceq \beta$, then $\gamma \in S'$. The final sections are defined similarly.

For any order relation \preceq on Φ^+ and for all $\beta \in \Phi^+$, we denote (β^{\preceq}) the \preceq -upper cone of β , i.e.,

$$(\beta^{\preceq}) = \{\gamma \in \Phi^+ \mid \beta \preceq \gamma\}.$$

Clearly, this is a dual order ideal, or filter, in the poset (Φ^+, \preceq) .

2.2. Basic lemmas on roots. We say that two roots are summable if their sum is a root. It is a basic fact that two roots with negative scalar product are summable and that the converse does not hold, in general.

Let \mathfrak{g} be a complex simple Lie algebra with root system Φ with respect to the Cartan subalgebra \mathfrak{h} (see e.g. [12, §18]). Thus, $\mathfrak{g} = (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha) \oplus \mathfrak{h}$, where \mathfrak{g}_α is the root space of α , for all $\alpha \in \Phi$, and $(\text{Span}_{\mathbb{C}}(\Phi)) = \mathfrak{h}^*$, the dual space of \mathfrak{h} . It is well known that if α and β are summable roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$, while if α and β are not summable and $\alpha \neq -\beta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.

Proposition 2.1. *Let Φ be any crystallographic root system and let $\beta_1, \beta_2, \beta_3 \in \Phi$ be such that $\beta_1 + \beta_2 + \beta_3 \in \Phi$ and $\beta_i \neq -\beta_j$ for all $i, j \in \{1, 2, 3\}$. Then at least two of the three sums $\beta_i + \beta_j$, with $i, j \in \{1, 2, 3\}$ and $i \neq j$, belong to Φ .*

Proof. Since $(\beta_1 + \beta_2 + \beta_3, \beta_1 + \beta_2 + \beta_3) > 0$, at least one of the scalar products $(\beta_1 + \beta_2 + \beta_3, \beta_i)$ with $1 \leq i \leq 3$ is strictly positive, whence $\beta_1 + \beta_2 + \beta_3 - \beta_i$ is a root. Assume for example $\beta_1 + \beta_2 \in \Phi$. Then, $[[\mathfrak{g}_{\beta_1}, \mathfrak{g}_{\beta_2}], \mathfrak{g}_{\beta_3}] \neq 0$, hence, by the Jacobi identity, at least one of $[[\mathfrak{g}_{\beta_1}, \mathfrak{g}_{\beta_3}], \mathfrak{g}_{\beta_2}]$ and $[\mathfrak{g}_{\beta_1}, [\mathfrak{g}_{\beta_2}, \mathfrak{g}_{\beta_3}]]$ is not 0. It follows that at least one of $\beta_1 + \beta_3$ and $\beta_2 + \beta_3$ is a root. \square

In the following Lemma we classify the Cartan integers of pairs of summable roots. The proof is an exercise and is omitted. The results are well known and will be used also without explicit reference to the lemma.

Lemma 2.2. *Assume $\beta, \gamma, \beta + \gamma \in \Phi$.*

(1) *If $|\beta| = |\gamma| = |\beta + \gamma|$, then $(\beta, \gamma^\vee) = -1$.*

(2) If $|\beta| = |\gamma| \neq |\beta + \gamma|$, then either $\frac{|\beta+\gamma|^2}{|\beta|^2} = 2$ and $(\beta, \gamma^\vee) = 0$, or $\frac{|\beta+\gamma|^2}{|\beta|^2} = 3$ and $(\beta, \gamma^\vee) = 1$. In any case, $|\beta| = |\gamma| < |\beta + \gamma|$.

(3) If $|\beta| < |\gamma|$, then $|\beta + \gamma| = |\beta|$, $(\gamma, \beta^\vee) = -\frac{|\gamma|^2}{|\beta|^2} \in \{-2, -3\}$, and $(\beta, \gamma^\vee) = -1$.

In particular, $(\beta, \gamma) \geq 0$ if and only if $|\beta| = |\gamma| < |\beta + \gamma|$.

2.3. Ad-nilpotent and abelian ideals. Let \mathfrak{g} be as in Subsection 2.2, \mathfrak{b} be the standard Borel subalgebra of \mathfrak{g} associated to Φ^+ , and \mathfrak{n} its nilpotent radical, i.e., $\mathfrak{b} = (\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha) \oplus \mathfrak{h}$ and $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$.

An *ad-nilpotent* ideal of \mathfrak{b} is a (nilpotent) ideal of \mathfrak{b} contained in \mathfrak{n} . It is clear that such an ideal is a sum of root spaces. For any $I \subseteq \Phi^+$, the sum of root spaces $\bigoplus_{\alpha \in I} \mathfrak{g}_\alpha$ is an ad-nilpotent ideal of \mathfrak{b} if and only if, for all $\alpha, \beta \in \Phi^+$, if $\alpha \in I$ and $\alpha \leq \beta$, then $\beta \in I$. A subset I of Φ^+ with this property is called an ad-nilpotent ideal of Φ^+ . Clearly, this is filter in (Φ, \leq) , i.e. a dual order ideal. It is easy to see that an abelian ideal of \mathfrak{b} must be ad-nilpotent. For any $I \subseteq \Phi^+$, the subspace $\bigoplus_{\alpha \in I} \mathfrak{g}_\alpha$ is an abelian ideal of \mathfrak{b} if and only if I is an ad-nilpotent ideal of Φ^+ with the further property that, for all $\alpha, \beta \in I$, $\alpha + \beta \notin \Phi$. Such an I is called an abelian ideal of Φ^+ . The abelian ideals of Φ^+ are studied in several papers, both for their implications in representation theory and for their proper algebraic-combinatorial interest. The main representation theoretic motivations can be found in [15, 16] (see also [8]); the basic algebraic-combinatorial results can be found [9], [10], [19], [20].

2.4. Abelian nilradicals. An ad-nilpotent ideal of Φ^+ is called *principal* if it has a minimum, i.e. if the corresponding \mathfrak{b} -ideal is principal. For all $\beta \in \Phi^+$, the upper \leq -cone of β , $(\beta^\leq) = \{\gamma \in \Phi^+ \mid \beta \leq \gamma\}$ is also called the principal ad-nilpotent ideal generated by β . It is clear that if $\beta \in \Phi^+$ is such that $c_\alpha(\beta) > \frac{m_\alpha}{2}$ for some $\alpha \in \Pi$, then (β^\leq) is abelian. In particular, this happens if β is a simple root of multiplicity 1 in Φ^+ . Indeed, the following well known result holds. The proof is brief, so we include it.

Proposition 2.3. *Let $S \subseteq \Pi$ and $I = \Phi^+ \setminus \Phi(S)$. Then I is an ad-nilpotent ideal. Moreover, I is abelian if and only if either $S = \Pi$, or $S = \Pi \setminus \{\alpha\}$ for a simple root α such that $m_\alpha = 1$. In this case, I is equal to (α^\leq) and is a maximal abelian ideal.*

Proof. It is clear that in any case I is an ad-nilpotent ideal. For $S = \Pi$ we obtain the empty root ideal. Let $\Pi \setminus S = \{\alpha\}$ with $\alpha \in \Pi$ and $m_\alpha = 1$. Then by definition we have $I = (\alpha^\leq)$, and it is clear that this is abelian. It remains to prove that it is maximal. If $S = \emptyset$, then $I = \Phi^+$ and the claim is obvious. If $S \neq \emptyset$, any nilpotent ideal J that strictly contains I also contains the highest root of $\Phi(S)$, and it is clear that this last root is summable to α . Hence, I is in any case a maximal abelian ideal.

Now, for all $\beta \in \Phi$, let $\text{ht}_{\Pi \setminus S}(\beta) = \sum_{\alpha \in \Pi \setminus S} c_\alpha(\beta)$. It is clear that the condition $\max\{\text{ht}_{\Pi \setminus S}(\beta) \mid \beta \in \Phi\} = 1$ is equivalent to $\Pi \setminus S = \{\alpha\}$ and $m_\alpha = 1$. In order to conclude the proof,

we assume $\max\{\text{ht}_{\Pi \setminus S}(\beta) \mid \beta \in \Phi\} > 1$ and prove that in this case I is not abelian. Let $\beta^* \in \text{Min}\{\beta \in \Phi \setminus \Phi(S) \mid \text{ht}_{\Pi \setminus S}(\beta) > 1\}$. Then $(\beta^*, \alpha') \leq 0$ for all $\alpha' \in S$ and, since $(\beta^*, \beta^*) > 0$, we must have $(\beta^*, \alpha) > 0$ for some $\alpha \in \Pi \setminus S$. For such an α , $\beta^* - \alpha$ is a root and belongs to I , since $\text{ht}_{\Pi \setminus S}(\beta^* - \alpha) > 0$. But $\beta^* - \alpha$ is summable to α , that also belongs to I , hence I is not abelian. \square

For each $S \subseteq \Pi$, $\bigoplus_{\alpha \in \Phi^+ \setminus \Phi(S)} \mathfrak{g}_\alpha$ is the nilradical (the largest nilpotent ideal) of the standard parabolic subalgebra associated to S (see [3, Ch. VIII, §3.4]). Hence, we will call the maximal abelian ideals (α^\leq) with $m_\alpha = 1$, together with the empty root ideal, *the abelian nilradicals*.

2.5. The faces of the root polytope. We recall some ideas and results from [5]. For all $\alpha \in \Pi$ and all $S \subseteq \Pi$, let

$$H_{\alpha, m_\alpha} = \{x \in E \mid (x, \check{\omega}_\alpha) = m_\alpha\}, \quad F_\alpha = H_{\alpha, m_\alpha} \cap \mathcal{P}, \quad F_S = \bigcap_{\alpha \in S} F_\alpha.$$

It is clear that the hyperplanes all H_{α, m_α} are supporting hyperplanes of \mathcal{P} , hence the F_α and F_S are faces of \mathcal{P} . We call them the *standard parabolic faces*. In fact, the set of all standard parabolic faces is a set of representatives of the orbits of the action of the Weyl group W on the set of all faces of \mathcal{P} [5].

For each standard parabolic face F , let

$$I_F = F \cap \Phi.$$

By definition, for each $S \subseteq \Pi$, I_{F_S} is the set of all roots β such that $c_\alpha(\beta) = m_\alpha$, for all $\alpha \in S$. It is easy to see that \mathcal{P} is the convex hull of the long roots (see e.g. [7]), hence the long roots in I_{F_S} are the vertexes of the face F_S .

For each $S \subseteq \Pi$, let

$$S^e = \{\theta\} \cup -S.$$

Moreover, let S_θ^e be the subset of S^e defined by the condition that $\Phi(S_\theta^e)$ is the irreducible component of $\Phi(S^e)$ containing θ . Finally, let $S_\theta = S_\theta^e \setminus \{\theta\}$.

It is clear that Π^e is the set of vertexes of the extended Dynkin graph of Φ with respect to the simple system $-\Pi$. We will call this extended Dynkin graph *the opposite* extended Dynkin graph (of Φ). Thus, by definition, the subgraph induced by S_θ^e in the opposite extended Dynkin graph is the connected component of θ .

It is immediate from the theory of affine root system, and very easy to see directly, that, for each *proper* subset S of Π , S^e is a simple system for the root subsystem $\Phi(S^e)$ that it generates.

The following proposition contains the preliminary results on the standard parabolic faces that we need. We note that the proposition also precises that the face F_S does not determine S . Indeed, by item (1) for all $S, S' \subseteq \Pi$, $F_S = F_{S'}$ if and only if $\Phi^+(\Pi \setminus S)_\theta^e =$

$\Phi^+((\Pi \setminus S')_\theta^e)$, i.e., F_S is uniquely determined by the irreducible component $\Phi^+((\Pi \setminus S)_\theta^e)$. In particular, the standard parabolic faces, and therefore the W -orbits of faces, are in bijection with the proper connected subgraphs of the opposite extended Dynkin graph that contains the vertex θ [21].

Proposition 2.4. [5] *Let $S \subseteq \Pi$, $S \neq \emptyset$.*

- (1) $I_{F_S} = \Phi^+((\Pi \setminus S)^e) \setminus \Phi(\Pi \setminus S) = \Phi^+((\Pi \setminus S)_\theta^e) \setminus \Phi((\Pi \setminus S)_\theta)$.
- (2) *Let μ_S be the highest root of $\Phi((\Pi \setminus S)_\theta^e)$, with respect to the simple system $(\Pi \setminus S)_\theta^e$. Then, I_{F_S} is the principal abelian ideal of Φ^+ generated by μ_S .*
- (3) $\dim(F_S) = |(\Pi \setminus S)_\theta|$.

By definition of I_{F_S} , statement (2) says that μ_S is the unique minimal root such that $c_\alpha(\mu_S) = m_\alpha$ for all $\alpha \in S$. Both (1) and (2) implies that we have $c_\alpha(\mu_S) < m_\alpha$ if and only if $\alpha \in (\Pi \setminus S)_\theta$. Hence, for all $\beta \in \Phi^+$, the condition $c_\alpha(\beta) = m_\alpha$ for all $\alpha \in S$ implies $c_\alpha(\beta) = m_\alpha$ also for all $\alpha \in \Pi \setminus (\Pi \setminus S)_\theta$, which in general is greater than S .

Remark 2.5. Let $\widetilde{\Phi}^+((\Pi \setminus S)^e)$ be the positive system of $\Phi((\Pi \setminus S)^e)$ relative to the simple system $(\Pi \setminus S)^e$. It is clear that for $S \neq \emptyset$ this positive system is different from $\Phi^+((\Pi \setminus S)^e)$, which is the intersection $\Phi(\Pi \setminus S) \cap \Phi^+$, by definition. However, $\Phi^+((\Pi \setminus S)^e) \setminus \Phi(\Pi \setminus S) = \widetilde{\Phi}^+((\Pi \setminus S)^e) \setminus \Phi(\Pi \setminus S)$. The same holds with $(\Pi \setminus S)_\theta$ in place of $\Pi \setminus S$. Therefore, in Proposition 2.4 (1) we may replace Φ^+ with $\widetilde{\Phi}^+$.

By the above remark, Proposition 2.4 (1) is equivalent to the following corollary.

Corollary 2.6. *The set I_{F_S} is the principal ideal generated by θ in the positive system $\widetilde{\Phi}^+((\Pi \setminus S)_\theta^e)$ of the irreducible root system $\Phi((\Pi \setminus S)_\theta^e)$.*

2.6. The order involution of face ideals. For all $w \in W$, let

$$N(w) = \{\beta \in \Phi^+ \mid w(\beta) \leq \underline{0}\}.$$

For all $S \subseteq \Pi$, let $w_{0,S}$ be the longest element in the standard parabolic subgroup of W generated by $\{s_\alpha \mid \alpha \in S\}$. It is well known that $w_{0,S}$ is an involution and is determined by the condition $N(w_{0,S}) = \Phi^+(S)$.

Proposition 2.7. *Let $S \subseteq \Pi$ and $w_S^* = w_{0,(\Pi \setminus S)}$. Then, the restriction of w_S^* to I_{F_S} is an anti-isomorphism of the poset (I_{F_S}, \leq) . In particular, w_S^* exchange θ and μ_S .*

Proof. By definition, $I_{F_S} = (\theta + L(\Phi(\Pi \setminus S))) \cap \Phi$ and, obviously, for all $\alpha \in \Pi \setminus S$, $s_\alpha(\theta) \in \theta + L(\Phi(\Pi \setminus S))$. Hence it is clear that w_S^* acts on I_{F_S} .

It remains to prove that w_S^* reverses the standard partial order on I_{F_S} . Let $\beta, \beta' \in I_{F_S}$ and $\beta < \beta'$. Then $\beta' - \beta \in L^+(\Phi(\Pi \setminus S))$, and since $w_S^*(\alpha) < \underline{0}$ for all $\alpha \in (\Pi \setminus S)$, $w_S^*(\beta') - w_S^*(\beta) = w_S^*(\beta' - \beta) \in -L^+(\Phi(\Pi \setminus S))$, i.e. $w_S^*(\beta') < w_S^*(\beta)$. \square

We note that, by Proposition 2.4(1), the above proposition holds also with $w_{0,(\Pi \setminus S)_\theta}$ in place of w_S^* . In particular, the restrictions of $w_{0,(\Pi \setminus S)_\theta}$ and of w_S^* on I_{F_S} coincide.

Definition 2.8. We call w_S^* the *face involution* of F_S and the restriction of w_S^* to I_{F_S} the *order involution* of I_{F_S} .

3. FACE IDEALS AND ABELIAN NILRADICALS

In this section we prove that the abelian nilradicals of Φ^+ are facet ideals and that all face ideals are abelian nilradicals in some irreducible subsystem of Φ .

By Proposition 2.4, the standard parabolic facets of \mathcal{P} are the faces of type F_α with $\alpha \in \Pi$ such that $\Phi((\Pi \setminus \{\alpha\})^e)$ is irreducible. Equivalently, a face F_α ($\alpha \in \Pi$) is a facet if and only if α is a leaf of the extended Dynkin diagram. In next results we prove that this happens, in particular, if $m_\alpha = 1$.

Proposition 3.1. *Each nonempty abelian nilradical of Φ^+ is a facet ideal.*

Proof. It is well known that if α is any simple root such that $m_\alpha = 1$, then the subgraph of the extended Dynkin graph obtained by removing α is isomorphic to the (ordinary) Dynkin graph of Φ [14]. In particular, $\Phi((\Pi \setminus \{\alpha\})^e)$ is irreducible. \square

We note that the fact that the simple roots α with $m_\alpha = 1$ are leafs of the extended Dynkin diagram is also a consequence of Proposition 2.7. Indeed, if $m_\alpha = 1$, then α is the minimum of I_{F_α} , hence, the order involution $w_{0,\Pi \setminus \{\alpha\}}$ maps α onto θ . Since it also maps $\Pi \setminus \{\alpha\}$ onto $-(\Pi \setminus \{\alpha\})$, it maps Π onto the nodes of the opposite extended Dynkin graph minus $-\alpha$.

It is clear that the converse of Proposition 3.1 is not true, however the following result holds.

Proposition 3.2. *Each face ideal in Φ^+ is an abelian nilradical of some irreducible root subsystem of Φ .*

Proof. By Corollary 2.6, any face ideal I_{F_S} ($S \subseteq \Pi$) is the principal ideal generated by θ in $\widetilde{\Phi}^+((\Pi \setminus S)^e)$.

It is clear that, for each $\beta \in \Phi((\Pi \setminus S)^e)$, in the expression of β as a linear combination of the base $(\Pi \setminus S)^e$, the coefficient of θ is at most 1. In other words, the multiplicity of θ , as a simple root in the positive system $\widetilde{\Phi}^+((\Pi \setminus S)^e)$, is 1. Hence, the principal ideal generated by θ in $\widetilde{\Phi}^+((\Pi \setminus S)^e)$ is an abelian nilradical. \square

Remark 3.3. Let $\alpha \in \Pi$ be such that F_α is a facet. By Proposition 2.4, I_{F_α} is also equal to $(\mu_{\{\alpha\}}^{\leq})$, where $\mu_{\{\alpha\}}$ is the unique root in Φ such that $c_\alpha(\mu_{\{\alpha\}}) = m_\alpha$ and $c_{\alpha'}(\mu_{\{\alpha\}}) < m_{\alpha'}$ for all $\alpha' \in \Pi \setminus \{\alpha\}$. By Proposition 2.7, the face involution $w_{\{\alpha\}}^*$ maps $(\Pi \setminus \{\alpha\})^e$ onto $\{\mu_{\{\alpha\}}\} \cup (\Pi \setminus \{\alpha\})$, therefore this last set is a simple system for $\Phi((\Pi \setminus \{\alpha\})^e)$. The positive system corresponding to it is the standard positive system $\Phi^+((\Pi \setminus \{\alpha\})^e)$ and,

clearly, has θ as its highest root. It is also clear that the multiplicity of $\mu_{\{\alpha\}}$, as a simple root in $\Phi^+(\Pi \setminus \{\alpha\})^e$, is 1. Thus, I_{F_α} is the abelian nilradical generated by $\mu_{\{\alpha\}}$ in the positive system $\Phi^+(\Pi \setminus \{\alpha\})^e$.

It is clear that the definition of ad-nilpotent and abelian ideals makes sense also in the reducible case. Let Ψ be any finite crystallographic root system, Ψ_1, \dots, Ψ_k be its irreducible components, Ψ_i^+ a positive system for Ψ_i , for $i = 1, \dots, k$, and $\Psi^+ = \Psi_1^+ \cup \dots \cup \Psi_k^+$. Then, by definition, I is an abelian ideal of Ψ^+ if and only if $I \cap \Psi_i^+$ is an abelian ideal of Ψ_i^+ for all $i \in \{1, \dots, k\}$. Moreover, I is an abelian nilradical of Ψ^+ if and only if $I \cap \Psi_i^+$ is an abelian nilradical of Ψ_i^+ for all $i \in \{1, \dots, k\}$. This means that $I \cap \Psi_i^+$ is either empty or a principal ideal generated by a simple root with multiplicity 1.

Lemma 3.4. *Let I be an abelian nilradical of Φ^+ , H a vector subspace in E , and $\Psi = H \cap \Phi$. Then $I \cap H$ is an abelian nilradical of Ψ^+ .*

Proof. Let $I = (\alpha^{\leq})$, with $\alpha \in \Pi$ and $m_\alpha = 1$. Let Ψ_1, \dots, Ψ_k be the irreducible components of Ψ , Π_{Ψ_i} be the simple system of Ψ_i^+ for $i = 1, \dots, k$, and let $\Pi_\Psi = \Pi_{\Psi_1} \cup \dots \cup \Pi_{\Psi_k}$, the simple system of Ψ^+ . Let $S_\alpha = \{\beta \in \Pi_\Psi \mid c_\alpha(\beta) = 1\}$. It is clear that if $\beta \in S_\alpha \cap \Pi_{\Psi_i}$, then β has multiplicity 1 in Ψ_i^+ . Moreover, since the sum of all roots in a fixed Π_{Ψ_i} is a root, for all $i \in \{1, \dots, k\}$, $S_\alpha \cap \Pi_{\Psi_i}$ contains at most one root. Hence, either $S_\alpha \cap \Pi_{\Psi_i} = \emptyset$, in which case $I \cap \Psi_i = \emptyset$, or $S_\alpha \cap \Pi_{\Psi_i} = \{\beta_i\}$ for a certain root β_i with multiplicity 1 in Ψ_i^+ . Then, clearly, $I \cap \Psi_i = (\beta_i^{\leq})$, hence $I \cap \Psi_i$ is an abelian nilradical of Ψ_i^+ . \square

4. CROSSING PAIRS

In this section we analyze the properties of *crossing pairs* contained in abelian ideals. In the simply laced case, many of the results that we are proving could be proved in a very simpler way.

Definition 4.1. Let $\beta_i, \gamma_i \in \Phi$, $i = 1, 2$, with $\beta_i \neq \gamma_j$ for all $i, j \in \{1, 2\}$. We say that $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ are *crossing pairs* if $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$. In this case we call the equality $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$ a *crossing relation*. We do not assume that $\beta_1 \neq \beta_2$ and $\gamma_1 \neq \gamma_2$, hence (at most) one of the pairs $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ may be a multiset of a single root with multiplicity 2.

Lemma 4.2. *Let I be an abelian ideal in Φ^+ .*

- (1) *If $\beta \in I \cap \Phi_s$, $x \in \Phi$, and $\beta + x \in \Phi$, then $x \in \Phi_s$.*
- (2) *If $\beta, \gamma \in I$ and $\beta - \gamma \in \Phi$, then $(\beta, \gamma) > 0$.*

Proof. (1) By contradiction, if $x \in \Phi_\ell^+$, then by Lemma 2.2(3) $(\beta, x) < 0$, hence $s_\beta(x) = x + \frac{|x|^2}{|\beta|^2}\beta \geq x + 2\beta$. It follows $x + 2\beta \in \Phi$, hence $x + 2\beta \in I$. This is impossible since $x + 2\beta = \beta + (x + \beta)$ and I is abelian.

(2) By Lemma 2.2, if $(\beta, \gamma) \leq 0$, then $\beta, \gamma \in \Phi_s$ and $\beta - \gamma \in \Phi_\ell$: by part (1) this is impossible. \square

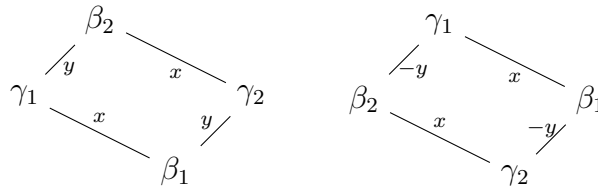
Proposition 4.3. *Let I be an abelian ideal in Φ^+ and $\{\beta_1, \beta_2\}, \{\gamma_1, \gamma_2\}$ be crossing pairs contained in I and such that $\beta_1 \neq \beta_2$. Then:*

- (1) *for all $i, j \in \{1, 2\}$ $(\beta_i, \gamma_j) > 0$, in particular $\beta_i - \gamma_j$ is a root;*
- (2) *either $\{\beta_1, \beta_2\}$, or $\{\gamma_1, \gamma_2\}$ is the pair of the minimum and maximum of $\{\beta_i, \gamma_i \mid i = 1, 2\}$.*
- (3) *$(\beta_1, \beta_2) = 0$ unless both of β_1, β_2 are short and γ_1, γ_2 have different lengths;*

Proof. (1) For $i \in \{1, 2\}$, $\beta_1 + \beta_2 - \gamma_i \in \Phi$, and since I is abelian, $\beta_1 + \beta_2 \notin \Phi$. By Lemma 2.1, we obtain $\beta_j - \gamma_i \in \Phi$ for $j \in \{1, 2\}$. By Lemma 4.2, it follows $(\beta_j, \gamma_i) > 0$ for $i, j \in \{1, 2\}$.

(2) We set $x = \gamma_1 - \beta_1 = \beta_2 - \gamma_2$ and $y = \gamma_2 - \beta_1 = \beta_2 - \gamma_1$. By (1), x and y are roots. If x and y are both positive or both negative, we directly obtain that $\{\beta_1, \beta_2\}$ is the set of the minimum and maximum of $\{\beta_i, \gamma_i \mid i = 1, 2\}$. Similarly, if one of x, y is positive and the other is negative, $\{\gamma_1, \gamma_2\}$ is the set of the minimum and maximum of $\{\beta_i, \gamma_i \mid i = 1, 2\}$. In the picture below the proof we illustrate the Hasse diagram of the quadruple $\{\beta_1, \beta_2, \gamma_1, \gamma_2\}$ in the cases $x, y > \underline{0}$ and $x > \underline{0}, y < \underline{0}$.

(3) We keep the notation of (2). We first assume that at least one of β_1, β_2 , is long. Let β_1 be long. Then, by (1) and Lemma 2.2, we have $(\beta_1^\vee, \gamma_2) = -(\beta_1^\vee, x) = 1$, hence $(\beta_1^\vee, \beta_2) = (\beta_1^\vee, \gamma_2 + x) = 0$. The case β_2 long is similar, so we assume that both β_1 and β_2 are short and $(\beta_1^\vee, \beta_2) \neq 0$. Then, $1 = (\beta_1^\vee, \beta_2) = (\beta_1^\vee, \beta_1) + (\beta_1^\vee, x) + (\beta_1^\vee, y) = 2 + (\beta_1^\vee, x) + (\beta_1^\vee, y)$. By Lemma 4.2, x and y are short, hence one of (β_1^\vee, x) and (β_1^\vee, y) is 0 and the other is -1 . By Lemma 2.2, this implies that one of γ_1 and γ_2 is long and the other is short. \square



By the above result, we may define the relations below.

Notation 4.4. We write $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$ for $\beta_1 < \gamma_i < \beta_2$ for both $i \in \{1, 2\}$.

Definition 4.5. We define the relations \lesssim and \sim on Φ^+ as follows:

$\beta_1 \lesssim \beta_2$ if and only if there exists $\gamma_1, \gamma_2 \in \Phi^+$ such that $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ are crossing pairs with $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$;

$\beta_1 \sim \beta_2$ if and only if either $\beta_1 \lesssim \beta_2$ or $\beta_2 \lesssim \beta_1$.

If $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ are crossing pairs with $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$, we also say that $\{\gamma_1, \gamma_2\}$ is a *middle pair between* β_1 and β_2 and that $\{\beta_1, \beta_2\}$ is a *raising pair through* γ_1 and γ_2 .

Next corollary precises the order relation among different raising pairs through a common middle pair and different middle pairs between a common raising pair.

Corollary 4.6. *Let I be an abelian ideal, $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ be crossing pairs in I with $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$.*

- (1) *If $\{\beta'_1, \beta'_2\}$ is any other raising pair through $\{\gamma_1, \gamma_2\}$, then either $\beta_1 < \beta'_1 < \beta'_2 < \beta_2$, or $\beta'_1 < \beta_1 < \beta_2 < \beta'_2$. Moreover, $\beta_i - \beta'_i \in \Phi$ for both $i = 1, 2$.*
- (2) *If $\{\gamma'_1, \gamma'_2\}$ is any other middle pair between $\{\beta_1, \beta_2\}$, then $\gamma_i - \gamma'_j \in \Phi$ for all $i, j \in \{1, 2\}$. Moreover, one of the following four cases occur: $\gamma'_i < \{\gamma_1, \gamma_2\} < \gamma'_j$, $\gamma_i < \{\gamma'_1, \gamma'_2\} < \gamma_j$ (with $\{i, j\} = \{1, 2\}$). In particular, there exists at most one incomparable middle pair between β_1 and β_2 .*

Proof. Under the assumption of (1), $\{\beta'_1, \beta'_2\}$ and $\{\beta_1, \beta_2\}$ are crossing pairs. Similarly, under the assumption of (2), $\{\gamma'_1, \gamma'_2\}$ and $\{\gamma_1, \gamma_2\}$ are crossing pairs. Hence the claim follows directly from Proposition 4.3. \square

In next Lemma, we see that the possible lengths of roots and root differences in a crossing pair are very limited.

Lemma 4.7. *Let I be an abelian ideal in Φ^+ , $\{\beta_1, \beta_2\}$, $\{\gamma_1, \gamma_2\}$ be crossing pairs contained in I , $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$, $x = \beta_2 - \gamma_1$, and $y = \beta_2 - \gamma_2$.*

- (1) *If x is long, then also y , β_1 , β_2 , γ_1 , γ_2 are long.*
- (2) *If any of x , y , β_1 , β_2 , γ_1 , γ_2 is short, then x and y are short and at most one of β_1 , β_2 , γ_1 , γ_2 is long, except when $\gamma_1 = \gamma_2$, in which case γ_1 is short and β_1, β_2 are long.*

Proof. We first prove that if one of x , y is short, then the other is short, too. Assume, for example, $x \in \Phi_s$. If $\beta_1 \in \Phi_s$, then $y \in \Phi_s$ by Lemma 4.2, hence let $\beta_1 \in \Phi_\ell$. In this case, by Lemma 2.2, $\beta_1 + x = \gamma_2 \in \Phi_s$, whence $y \in \Phi_s$ by Lemma 4.2.

Hence, x, y are either both short, or both long. In order to prove (1), it remains to check that if $x, y \in \Phi_\ell$, then $\beta_i, \gamma_j \in \Phi_\ell$ for $i = 1, 2$. This follows directly from Lemma 4.2 for β_1 , γ_1 , and γ_2 . For β_2 it follows from Lemma 2.2, since $\beta_2 = \gamma_1 + x$.

It remains to conclude the proof of (2). By (1), if any of x , y , β_1 , β_2 , γ_1 , γ_2 belongs to Φ_s , then $x, y \in \Phi_s$. In this case, assume $\beta_i \in \Phi_\ell$ for a certain $i \in \{1, 2\}$, and let $\{i'\} = \{1, 2\} \setminus \{i\}$. For $j \in \{1, 2\}$, $\beta_i - \gamma_j \in \{\pm x, \pm y\}$, hence by Lemma 2.2, $\gamma_j \in \Phi_s$. If also $\beta_{i'} \in \Phi_\ell$ then, by Lemma 2.2, $(\gamma_i, x) = (\gamma_i, y) = 0$ for both $i = 1, 2$, hence $(\gamma_2, \gamma_1^\vee) = (\gamma_1 + x - y, \gamma_1^\vee) = 2$, which implies $\gamma_1 = \gamma_2$, since $|\gamma_1| = |\gamma_2|$. Conversely, if $\gamma_1 = \gamma_2$, then we get $-\beta_{i'} = \beta_i - 2\gamma_1 = s_{\gamma_1}(\beta_i)$, where s_{γ_1} is the reflection with respect to γ_1 , hence $|\beta_{i'}| = |\beta_i|$.

By a similar argument, taking into account that $\beta_1 \neq \beta_2$, we obtain that if one of γ_1, γ_2 is long, all the remaining roots in the crossing pairs are short. \square

In next proposition, we prove that, for comparable roots β_1 and β_2 in an abelian ideal I , if $\beta_1 - \beta_2$ is not a root, then $\beta_1 \sim \beta_2$. Moreover, we analyze when the reverse implication holds. In particular, we see that this happens when β_1 and β_2 are long roots, hence in the simply laced case, i.e., in this case we have $\beta_1 \sim \beta_2$ if and only if $\beta_1 - \beta_2 \notin \Phi_\ell$.

Proposition 4.8. *Let I be an abelian ideal in Φ^+ and $\beta_1, \beta_2 \in I$.*

- (1) *If $\beta_1 < \beta_2$ and $\beta_2 - \beta_1 \notin \Phi$, then $\beta_1 \lesssim \beta_2$.*
- (2) *If $\beta_1 \lesssim \beta_2$, $\{\beta_1, \beta_2\} \subseteq \Phi_s$ and there exists a middle pair $\{\gamma_1, \gamma_2\}$ between β_1, β_2 such that $\gamma_1 \in \Phi_s$ and $\gamma_2 \in \Phi_\ell$, then $\beta_2 - \beta_1 \in \Phi_s$.*
- (3) *If $\beta_1 \lesssim \beta_2$, then $\beta_2 - \beta_1 \notin \Phi$ if and only if either of the following conditions is satisfied:*
 - (a) *at least one of β_1, β_2 is long,*
 - (b) *$\{\beta_1, \beta_2\} \subseteq \Phi_s$ and there exists a middle pair $\{\gamma_1, \gamma_2\} \subseteq \Phi_s$ between β_1, β_2 .*

Proof. (1) Let $\beta_1 < \beta_2$ and $\beta_2 - \beta_1 \notin \Phi$. By definition, $\beta_2 - \beta_1$ is a sum of positive roots. Let

$$k = \min\{h \in \mathbb{N} \mid \exists \eta_1, \dots, \eta_h \in \Phi^+ \text{ such that } \beta_2 - \beta_1 = \eta_1 + \dots + \eta_h\},$$

and $\eta_1, \dots, \eta_k \in \Phi^+$ be such that $\beta_2 = \beta_1 + \eta_1 + \dots + \eta_k$. By assumption, $k \geq 2$ and no sum $\sum_{j=1}^h \eta_{i_j}$ with $1 \leq i_j \leq k$ and $h > 1$ is a root. Clearly, at least one among (β_2, β_1) , (β_2, η_i) with $1 \leq i \leq k$, must be strictly positive. Now $\beta_2 - \beta_1 \notin \Phi$ by assumption, and also $\beta_1 + \beta_2 \notin \Phi$, since I is abelian, hence $(\beta_1, \beta_2) = 0$. Therefore $(\beta_2, \eta_i) > 0$ for some $i \in \{1, \dots, k\}$. We may assume $(\beta_2, \eta_k) > 0$, so that $\beta_2 - \eta_k = \beta_1 + \eta_1 + \dots + \eta_{k-1} \in \Phi$. Let $\gamma_i = \beta_1 + \sum_{1 \leq j \leq i} \eta_j$: iterating the above argument, we may assume $\gamma_i \in \Phi$ for all $i \in \{0, \dots, k\}$. Since $\eta_i + \eta_j \notin \Phi$, for $1 \leq i, j \leq k$, by Proposition 2.1 applied to any sum $\gamma_i + \eta_{i+1} + \eta_{i+2}$, we get that both $\gamma_i + \eta_{i+1}$ and $\gamma_i + \eta_{i+2}$ belong to Φ , for $0 \leq i \leq k-2$. It follows easily that, for any rearrangement η'_1, \dots, η'_k of η_1, \dots, η_k , $\gamma'_i = \beta_1 + \sum_{1 \leq j \leq i} \eta'_j$ is a root, for $0 \leq i \leq k$. In particular, $\beta_1 + \eta_1$ and $\beta_2 - \eta_1$ are roots, both different from β_1 and β_2 , hence $\beta_1 + \beta_2 = (\beta_1 + \eta_1) + (\beta_2 - \eta_1)$ is a crossing relation.

(2) Let $\beta_1, \beta_2, \gamma_1 \in \Phi_s$ and $\gamma_2 \in \Phi_\ell$, $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$, $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$, $x = \beta_2 - \gamma_1 = \gamma_2 - \beta_1$, and $y = \beta_2 - \gamma_2 = \gamma_1 - \beta_1$. Then, by Lemmas 4.2(2) and 2.2, we obtain $(\beta_2, \beta_1^\vee) = (\gamma_2 + y, \beta_1^\vee) = 2 + (y, \beta_1^\vee) \geq 1$ and hence $(\beta_2, \beta_1^\vee) = 1$ and $\beta_2 - \beta_1 \in \Phi_s$.

(3) Let $\beta_1, \beta_2, \gamma_1, \gamma_2 \in \Phi^+$ be such that $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$, $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$, $x = \beta_2 - \gamma_1$, and $y = \beta_2 - \gamma_2$. By Lemma 4.7(2), if neither (a) nor (b) hold, then we are in the case of item (2), hence $\beta_1 - \beta_2 \in \Phi$. It remains to prove the converse.

Let $\beta_2 \in \Phi_\ell$. Then by Lemma 2.2 $(x, \beta_2^\vee) = (y, \beta_2^\vee) = 1$, hence

$$(*) \quad 2 = (\beta_2, \beta_2^\vee) = (\beta_1 + x + y, \beta_2^\vee) = (\beta_1, \beta_2^\vee) + 2.$$

It follows $(\beta_1, \beta_2^\vee) = 0$, and $\beta_2 - \beta_1 \notin \Phi$. The case $\beta_1 \in \Phi_\ell$ is similar.

If $\beta_1, \beta_2, \gamma_1, \gamma_2 \in \Phi_s$, then by Lemma 4.2 also $x, y \in \Phi_s$, hence equalities $(*)$ still hold and we can argue as above. \square

Definition 4.9. For any $S \subseteq \Phi^+$, we say that S is reduced if, for all $\beta, \beta' \in S$, $\beta \not\prec \beta'$.

For all $\beta \in \Phi^+$ we set

$$\text{Red}(\beta) = \{\beta' \in \Phi^+ \mid \beta \neq \beta' \text{ and } \beta \approx \beta'\}, \quad \text{Red}(\beta)^\leq = \text{Red}(\beta) \cap (\beta^\leq).$$

Remark 4.10. By Proposition 4.8 and Lemma 4.2,

$$\text{Red}(\beta)^\leq \subseteq \{\eta \in (\beta^\leq) \mid \eta - \beta \in \Phi^+\} = \{\eta \in (\beta^\leq) \mid (\eta, \beta) > 0\}.$$

Moreover, in the simply laced case, the inclusion is an equality. In general, the inclusion is proper. As an example, in type C_n , if we number the simple roots as in [2], and take $\beta_1 = \alpha_n + \alpha_{n-1}, \beta_2 = \alpha_n + 2\alpha_{n-1} + \alpha_{n-2}, \gamma_1 = \alpha_n + 2\alpha_{n-1}, \gamma_2 = \alpha_n + \alpha_{n-1} + \alpha_{n-2}$, we have: $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$, hence $\beta_1 \lesssim \beta_2$, but $\beta_2 - \beta_1 = \alpha_{n-1} + \alpha_{n-2} \in \Phi_s$.

5. TRIANGULATION ORDERS

In this section we define some special orderings of abelian ideals, which we call triangulation orders, and prove that all facet ideals have a triangulation order. Throughout the section, let I be an abelian ideal of Φ^+ such that $\text{rk}(I) = n$.

Definition 5.1. Let $J \subseteq I$. We say that J is bipartite if it has an initial section J_i , and a final section J_f such that

- (1) $J = J_i \cup J_f$;
- (2) for all $\beta_1 \in J_i \setminus J_f$ and $\beta_2 \in J_f \setminus J_i$, we have $\beta_1 \lesssim \beta_2$;
- (3) there exists a hyperplane H in E such that $J_i \cap J_f \subseteq H$ and H strictly separates $J_i \setminus J_f$ from $J_i \setminus J_f$.

If the above conditions hold, we say that $\{J_i, J_f\}$ is a bipartition of J . If, moreover, J_i and J_f are both proper subsets of J , we say that $\{J_i, J_f\}$ is a proper bipartition. A hyperplane H as in (3) is called a separating hyperplane, for the bipartition $\{J_i, J_f\}$ of J .

Note that, by definition, if J has a proper bipartition, then it has at least two elements. If J is also saturated, it has at least three. From the definition it also follows directly that if $\{J_i, J_f\}$ is a bipartition of J , then $J_i \setminus J_f$ and $J_f \setminus J_i$ are an initial and a final section of J . Indeed, for example, if $\beta_1 \in J_i \setminus J_f$ and $\beta \in J \setminus (J_i \setminus J_f)$ we have $\beta \not\prec \beta_1$, because $J \setminus (J_i \setminus J_f) = J_f$ and J_f is a final section. Moreover, it is clear that $(J_i \setminus J_f) \leq (J_f \setminus J_i)$. Finally, we note that if J is saturated, also J_i and J_f are saturated.

Definition 5.2. For each subset S of Φ^+ , we define the restricted relations \lesssim_S and \sim_S on S as follows. For all $\beta_1, \beta_2 \in S$: (1) $\beta_1 \lesssim_S \beta_2$ if and only if there exists a middle pair $\{\gamma_1, \gamma_2\}$ between β_1 and β_2 contained in S ; (2) $\beta_1 \sim_S \beta_2$ if and only if either $\beta_1 \lesssim_S \beta_2$, or $\beta_2 \lesssim_S \beta_1$. We say that S is \sim -closed if, for all $\beta_1, \beta_2 \in S \cap \Psi^+$, $\beta_1 \lesssim \beta_2$ implies $\beta_1 \lesssim_S \beta_2$.

It is clear that, for any $S \subseteq \Phi^+$, the relation $\beta_1 \lesssim_S \beta_2$ implies $\beta_1 \lesssim \beta_2$, while the converse, in general, does not hold. Hence, if S is \sim -closed, for all $\beta_1, \beta_2 \in S$ we have $\beta_1 \sim \beta_2$ if and only if $\beta_1 \sim_S \beta_2$.

The first of following lemmas is clear, hence we omit the proof.

Lemma 5.3. *Let $S \subseteq \Phi^+$. If S is saturated, then S is \sim -closed.*

Lemma 5.4. *Let I be an abelian ideal in Φ , Ψ a root subsystem of Φ , and Ψ_1, \dots, Ψ_k be the irreducible components of Ψ . If $I \cap \Psi$ is \sim -closed, then for all $\beta_1, \beta_2 \in I \cap \Psi$, $\beta_1 \lesssim \beta_2$ if and only if there exists $i \in \{1, \dots, k\}$ such that $\beta_1, \beta_2 \in \Psi_i$ and $\beta_1 \lesssim_{I \cap \Psi_i} \beta_2$.*

Proof. Let $\beta_1, \beta_2 \in I \cap \Psi$ and $\beta_1 \lesssim \beta_2$. If $I \cap \Psi$ is \sim -closed, there exists a middle pair $\{\gamma_1, \gamma_2\} \subseteq I \cap \Psi$, between β_1 and β_2 . By Proposition 4.3, $(\beta_i, \gamma_j) > 0$ for all $i, j \in \{1, 2\}$, hence all of β_i and γ_i belong to the same irreducible component of Ψ . \square

Lemma 5.5. *Let I be an abelian nilradical of Φ^+ , Ψ a parabolic subsystem of Φ , and Π_Ψ the simple system of Ψ^+ . Assume that: (1) $\Pi_\Psi \setminus I \subseteq \Pi$; (2) each maximal root in $I \cap \Psi$ is comparable with at most one root in $\Pi_\Psi \cap I$. Then $I \cap \Psi$ is saturated, hence \sim -closed.*

Proof. As seen in the proof of Lemma 3.4, $I \cap \Psi = \bigcup_{\beta \in \Pi_\Psi \cap I} (\beta^\preceq)$. Moreover, different elements in $\Pi_\Psi \cap I$ belong to different irreducible components of Ψ .

The maximal elements in $I \cap \Psi$ are clearly the highest roots of the irreducible components of Ψ that have nonempty intersection with I , hence, condition (2) implies that, for any pair of irreducible components Ψ_1 and Ψ_2 of Ψ , $I \cap \Psi_1$ and $I \cap \Psi_2$ are element-wise incomparable with respect to the standard partial order \leq of Φ . Therefore, $I \cap \Psi$ is saturated if and only if, for each irreducible component Ψ_1 of Ψ , $I \cap \Psi_1$ is saturated.

Let Ψ_1 be a fixed irreducible component of Ψ such that $I \cap \Psi_1 \neq \emptyset$, $\Pi_1 = \Pi_\Psi \cap \Psi_1$, and $\beta_1, \beta_2 \in \Psi_1$. Then, $\beta_1 - \beta_2$ is a \mathbb{Z} -linear combination of roots in $\Pi_1 \setminus I$. By condition (1), this is contained in Π , which is \mathbb{Z} -basis of $L(\Phi)$, hence, if $\beta_1 - \beta_2 \in L^+(\Phi)$, we obtain that $\beta_1 - \beta_2 \in L^+(\Psi_1)$. In this case, since Ψ , and hence Ψ_1 , is parabolic, we have also that all roots between β_1 and β_2 belong to Ψ_1 , hence to $I \cap \Psi_1$.

This proves that $I \cap \Psi$ is saturated. By Lemma 5.3, it is also \sim -closed. \square

Definition 5.6. Let $J \subseteq I$, and $\beta \in J$. We say that β is a detachable element in J if the following conditions hold:

- (1) β is an extremal element of J with respect to the standard partial order;
- (2) there exists a hyperplane H such that:

- (a) $\text{Red}(\beta) \cap J = J \cap H$ and H strictly separates β from $J \setminus (\{\beta\} \cup \text{Red}(\beta))$;
- (b) $I \cap H$ is \sim closed.

We call such a hyperplane H a detaching hyperplane for β in J .

Remark 5.7. Let β be detachable in J , H be a detaching hyperplane, and $J_\beta = \{\beta\} \cup (J \cap H)$. Then, $J_\beta = \{\beta\} \cup (J \cap \text{Red}(\beta))$ and it is immediate from Definition 5.1 that $\{J_\beta, J \setminus \{\beta\}\}$ is a bipartition of J .

Lemma 5.8. *Let $\beta \in I \cap \Phi_\ell$. Then there exist a hyperplane H such that $I \cap H = \text{Red}(\beta)$, H strictly separates β from $I \setminus (\text{Red}(\beta) \cup \{\beta\})$, and $I \cap H$ is \sim closed. In particular, for all $J \subseteq I$ such that $\beta = \min J$ or $\beta = \max J$, β is detachable in J and H is a detaching hyperplane for β in J .*

Proof. Let α_I be the (unique) simple root such that $I = \{\gamma \in \Phi \mid c_{\alpha_I}(\gamma) = m_{\alpha_I}\}$. By Proposition 4.8, for all $\gamma \in J \setminus \{\beta\}$ we have $\beta \approx \gamma$ if and only if $\beta - \gamma \in \Phi$. Since $\beta \in \Phi_\ell$, this condition is equivalent to $(\beta^\vee, \gamma) = 1$. Recall that $\tilde{\omega}_{\alpha_I}$ is the fundamental coweight such that $(\alpha_I, \tilde{\omega}_{\alpha_I}) = 1$, and let $\nu = m_{\alpha_I}\beta^\vee - \tilde{\omega}_{\alpha_I}$, $H = \nu^\perp$. Then, $(\nu, \beta) = m_{\alpha_I}$, and $(\nu, \gamma) = 0$ for all γ in J such that $(\beta^\vee, \gamma) = 1$. Since I is abelian, for all other $\gamma \in I \setminus \{\beta\}$ we have $(\beta^\vee, \gamma) = 0$, hence $(\nu, \gamma) = -m_{\alpha_I}$. Thus we have proved that $I \cap H = \text{Red}(\beta)$, and H strictly separates β from $I \setminus (\text{Red}(\beta) \cup \{\beta\})$.

It remains to prove that $I \cap H$ is \sim closed. Let $\beta_1, \beta_2 \in \Phi^+ \cap H$, $\beta_1 \sim \beta_2$, and $\{\gamma_1, \gamma_2\}$ be a middle pair between β_1 and β_2 . Then $(\gamma_1 + \gamma_2, \beta^\vee) = (\beta_1 + \beta_2, \beta^\vee) = 2$. Since β is long, this forces $(\gamma_1, \beta^\vee) = (\gamma_2, \beta^\vee) = 1$, hence $\{\gamma_1, \gamma_2\} \subseteq I \cap H$, and $\beta_1 \sim_{I \cap H} \beta_2$, as claimed. \square

Definition 5.9. Let \preceq be a total order relation on I ,

$$S_{I, \preceq} = \{\beta \in I \mid \text{rk}(\beta^{\preceq}) = n\}.$$

We say that \preceq is a *triangulation order* if the following conditions hold:

- (1) $I \cap \text{Span}(I \setminus S_{I, \preceq})$ is saturated;
- (2) for each $\beta \in S_{I, \preceq}$, (β^{\preceq}) is saturated and either of the following conditions holds:
 - (a) β is detachable in (β^{\preceq}) ,
 - (b) (β^{\preceq}) has a bipartition $\{J_i, J_f\}$ such that, for both $J = J_i$ and J_f , β is detachable in J , and there exist a detaching hyperplane H_J , for β in J , such that $(\beta^{\preceq}) \cap H_J \subseteq \text{Red}(\beta)$.

Remark 5.10. (1) It is clear that, for any total ordering \preceq on I , the subset $S_{I, \preceq}$ is an initial section of the ordered set (I, \preceq) . Moreover, $\text{rk}(I \setminus S_{I, \preceq}) < n$.

(2) It may happen that $I \setminus S_{I, \preceq}$ be properly contained in $I \cap \text{Span}(I \setminus S_{I, \preceq})$. In fact, this happens for a triangulation order that we will construct for type E_7 .

- (3) The above definition does not contain any condition on the restriction of \preceq to $I \setminus S_{I, \preceq}$. Hence, if \preceq is a triangulation order, any other total order \preceq' such that $S_{I, \preceq'} = S_{I, \preceq}$, and \preceq and \preceq' coincide on the initial section $S_{I, \preceq}$, is a triangulation order, too.

We will prove the existence of triangulation orders for all facet ideals. The proof requires a case by case analysis. By Proposition 3.2, we may restrict the analysis to the abelian nilradicals.

Definition 5.11. We say that the facet ideal I of Φ^+ is an abelian nilradical of type $X_{n,k}$, and we write $I \cong X_{n,k}$, if there exists an irreducible root subsystem Ψ of Φ and a positive system $\tilde{\Psi}^+$ of Ψ such that I is an abelian nilradical in $\tilde{\Psi}^+$ and:

- (1) Ψ is of type X_n ;
- (2) if $\{\alpha'_1, \dots, \alpha'_n\}$ is a simple system of $\tilde{\Psi}^+$, numbered according to Bourbaki's conventions [2], then I is the principal ideal generated by α'_k in $\tilde{\Psi}^+$.

It is implicit in the definition that the above α'_k has multiplicity 1 in Ψ .

We note that the type of a facet ideal may be not unique, if the root system Ψ has nontrivial Dynkin diagram automorphisms. We identify the types $X_{n,k}$ and $X_{n,k'}$ if there exists a diagram automorphism that maps α_k into $\alpha_{k'}$. By a direct inspection of the highest root in all root types, we see that the possible types of abelian nilradicals type, in an irreducible root system of rank n , are the following:

$$A_{n,k} \text{ for } k = 1, \dots, n, \quad B_{n,1}, \quad C_{n,n}, \quad D_{n,k} \text{ for } k = 1, n-1, n, \quad E_{6,1}, \quad E_{6,6}, \quad E_{7,7}.$$

Among them, we have the identifications: $A_{n,k} = A_{n,k'}$ for $k + k' = n + 1$; $D_{n,n-1} = D_{n,n}$ for all $n \geq 4$ and $D_{n,1} = D_{n,n-1} = D_{n,n}$ for $n = 4$; $E_{6,1} = E_{6,6}$.

By Proposition 3.2, the facet ideals that are not abelian nilradicals of Φ^+ are in any case abelian nilradicals of some type. Their type $X_{n,k}$ is explicitly obtained as follows.

Let α_i be a leaf in the extended Dynkin diagram of Φ , so that $I_{F_{\alpha_i}}$ is a facet ideal of Φ^+ . By Corollary 2.6, the Dynkin diagram obtained by removing α_i from the extended Dynkin diagram of Φ , gives the root type X_n . The position of $-\theta$ in this diagram gives the index k of the abelian nilradical type $X_{n,k}$. Below, we write the resulting type for the facet ideals that are not abelian nilradicals of Φ^+ itself. If the root type of Φ is Y_n , we write $I_F(Y_n, \alpha_i)$ in place of $I_{F_{\alpha_i}}$.

$$\begin{aligned} I_F(B_n, \alpha_n) &\cong D_{n,n}, & I_F(F_4, \alpha_4) &\cong B_{4,1}, & I_F(E_7, \alpha_2) &\cong A_{7,1}, \\ I_F(E_8, \alpha_1) &\cong D_{8,1}, & I_F(E_8, \alpha_2) &\cong A_{8,1}. \end{aligned}$$

In proving next proposition, we will consider, case by case, the seven possible distinct sporadic or classes of abelian nilradical types. The main points of the proof are illustrated in Figures 1-9. We first give some explanation of these figures. We may arrange the roots of any facet ideal I in a matrix $(\beta_{i,j})$, in such a way that adjacent entries differ by a

simple root. The label i on a certain edge means that the difference between its vertexes is the simple root α_i . We choose the matrix arrangement of roots so that the standard partial order is compatible with the reverse lexicographic order of row and column indexes, starting from $\beta_{1,1} = \theta$. In this way, the matrix yields a Hasse diagram of I in which the order ascends toward northwest. We note that this condition do not determine a unique possibility. The figures illustrate the proof on such Hasse diagrams for all the abelian nilradicals.

Proposition 5.12. *Each facet ideal has a triangulation order.*

Proof. By the above discussion, we may assume that I is an abelian nilradical of Φ^+ .

By Remark 5.10, it suffices to define a subset $S_{I,\preccurlyeq}$ of I and a partial order \preccurlyeq on I that is total on $S_{I,\preccurlyeq}$ and has $S_{I,\preccurlyeq}$ as an initial section, in such a way that conditions (1) and (2) of Definition 5.9 are satisfied. We also require $\text{rk}(I \setminus S_{I,\preccurlyeq}) = n - 1$, and $\text{rk}(\beta^\preccurlyeq) = n$ for each $\beta \in S_{I,\preccurlyeq}$, in order that $S_{I,\preccurlyeq} = \{\beta \in I \mid \text{rk}(\beta^\preccurlyeq) = n\}$.

Henceforward, we write S_I in place of $S_{I,\preccurlyeq}$ and we intend that S_I is an initial section of \preccurlyeq . In all cases, we define the restriction (S_I, \preccurlyeq) as a sequence $(\beta_1, \dots, \beta_k)$ such that, for $i = 1, \dots, k$, β_i is an extremal element in $I \setminus \{\beta_j \mid j < i\}$, with respect to the standard partial order. This ensures that (β_i^\preccurlyeq) is saturated. Therefore, in order to prove condition (2), it will remain to prove that either condition (a), or (b) holds for all β_i .

If β_i is long and $\beta_i = \min(\beta_i^\preccurlyeq)$, or $\beta_i = \max(\beta_i^\preccurlyeq)$ (with respect to the standard partial order), then β_i is detachable in (β_i^\preccurlyeq) by Lemma 5.8, and we have nothing to prove. In the remaining cases, we will directly prove that (a) or (b) holds.

Finally, since we take β_i extremal in (β_i^\preccurlyeq) by construction, in order to prove that β_i is detachable in (β_i^\preccurlyeq) , or in a subset of its, it will suffice to check condition (2) in Definition 5.6.

Now we can give the details of the proof for each abelian nilradical. Throughout the rest of the proof, we use the following notation: for $h, k \in \{1, \dots, n\}$, $\omega_h = \check{\omega}_{\alpha_h}$, $\alpha_{[h,k]} = \sum_{h \leq i \leq k} \alpha_i$; for $S \subseteq \{1, \dots, n\}$, $\alpha_S = \sum_{i \in S} \alpha_i$.

A. $I \cong A_{n,k}$, $[\frac{n}{2}] < k \leq n$. We define $(S_I, \preccurlyeq) = (\alpha_{[k,j]} \mid j = k, \dots, n)$. It is easily seen that $I \setminus S_I$ is the type $A_{n-1,k-1}$ abelian nilradical generated by $\alpha_k + \alpha_{k-1}$ in the root subsystem that has $\{\alpha_{k-1} + \alpha_k\} \cup \Pi \setminus \{\alpha_{k-1}, \alpha_k\}$ as a simple system. This implies that $\text{rk}(I \setminus S_I) = n - 1$ and, by Lemma 5.5, that $I \cap \text{Span}(I \setminus S_I)$ is \sim closed.

For all $\beta \in S_I$, $\text{rk}(\beta^\preccurlyeq) = n$ and β is detachable in (β^\preccurlyeq) . Indeed, for $\beta = \alpha_{[k,j]}$, with $j \in [k, n]$, let $H = (\check{\omega}_k - \check{\omega}_{k-1} - \check{\omega}_{j+1})^\perp$ (where $\check{\omega}_{n+1} = \underline{0}$). Then, if $j < n$, the simple system of $(\Phi \cap H)^+$ is $\{\alpha_{[k-1,k]}, \alpha_{[k,j+1]}\} \cup \Pi \setminus \{\alpha_{k-1}, \alpha_k, \alpha_{j+1}\}$, while the maximal roots are $\alpha_{[1,j]}$ and $\alpha_{[k,n]}$. If $j = n$, the simple system is $\{\alpha_{[k-1,k]}\} \cup \Pi \setminus \{\alpha_{k-1}, \alpha_k\}$, and $\Phi \cap H$ is irreducible. By Lemma 5.5, we obtain that $I \cap H$ is \sim closed. It remains to check that condition (2a) of Definition 5.6 hold. Let $\gamma \in (\beta^\preccurlyeq)$. If $\gamma \in H$, either γ and β are

incomparable for the standard partial order, or $\gamma - \beta \in \Phi^+$, while, if $\gamma \notin H$, we have $\gamma \geq \beta$ and $(\gamma, \beta) = 0$. By Proposition 4.8, we obtain that $\gamma \sim \beta$ if and only if $\gamma \notin H$, which is the claim.

C. $I \cong C_{n,n}$. We define $(S_I, \preceq) = (\alpha_{[j,n]} | j = n, n-1, \dots, 1)$. It is easy to see that $I \setminus S_I$ is the type $C_{n-1,n-1}$ abelian nilradical generated by $\alpha_n + 2\alpha_{n-1}$ in the root subsystem that has $\{\alpha_n + 2\alpha_{n-1}\} \cup \Pi \setminus \{\alpha_{n-1}, \alpha_n\}$ as a simple system. Hence, $\text{rk}(I \setminus S_I) = n-1$ and, by Lemma 5.5, $I \cap \text{Span}(I \setminus S_I)$ is \sim -closed.

For all $\beta \in S_I$, $\text{rk}(\beta^\preceq) = n$ and β is detachable in (β^\preceq) . Indeed, for $\beta = \alpha_{[j,n]}$, $j \in [1, n]$, we take $H = (2\check{\omega}_n - \check{\omega}_{n-1} - \check{\omega}_{j-1})^\perp$. Then, the simple system of $(\Phi \cap H)^+$ is $\{\alpha_{[j-1,n]}, \alpha_n + 2\alpha_{n-1}\} \cup \Pi \setminus \{\alpha_n, \alpha_{n-1}, \alpha_{j-1}\}$ for $j < n$, and $\{\alpha_n + \alpha_{n-1}\} \cup \Pi \setminus \{\alpha_n, \alpha_{n-1}\}$ for $j = n$. For $j < n$, the maximal roots of $(\Phi \cap H)^+$ are $\alpha_{[j,n]} + \alpha_{[j,n-1]}$ and $\alpha_{[1,n]}$. For $j = n$, $\Phi \cap H$ is irreducible. It follows that $I \cap H$ is \sim closed, by Lemma 5.5. If $\gamma \in I$, then $\gamma = \alpha_{[h,n]} + \alpha_{[k,n-1]}$ for some $1 \leq h \leq k \leq n$. Hence, $\gamma \in H$ if and only if either $h \leq j-1$ and $k = n$, or $j \leq h \leq k \leq n-1$. In these cases, either γ and β are incomparable for the standard partial order, or $\gamma - \beta \in \Phi^+$, and all γ' such that $\gamma < \gamma' < \beta$ are short roots. In any case, $\gamma \not\sim \beta$ by Proposition 4.8(3). If $\gamma \in (\beta^\preceq) \setminus H$, we have $\gamma = \alpha_{[h,n]} + \alpha_{[k,n-1]}$ with $h \leq j-1 \leq k \leq n-1$, hence $\beta + \alpha_{[k,n-1]} \in \Phi$ and we obtain a crossing relation. It follows that H satisfies the conditions of Definition 5.6.

B and D1. $I \cong B_{n,1}$, or $I \cong D_{n,1}$. We define $(S_I, \preceq) = (\alpha_1, \theta)$. It is easy to see that $I \setminus S_I$ is the type $B_{n-1,1}$, or $D_{n-1,1}$, abelian nilradical generated by $\alpha_1 + \alpha_2$ in the subsystem whose simple system is $\{\alpha_1 + \alpha_2\} \cup \Pi \setminus \{\alpha_1, \alpha_2\}$. Hence, $\text{rk}(I \setminus S_I) = n-1$ and, by Lemma 5.5, $I \cap \text{Span}(I \setminus S_I)$ is \sim -closed. For all $\beta \in S_I$, $\text{rk}(\beta^\preceq) = n$ and β is detachable in (β^\preceq) by Lemma 5.8.

Dn. $I \cong D_{n,n}$. We define $(S_I, \preceq) = (\hat{\alpha}_{[j,n]} | j = n, n-2, \dots, 1)$, where $\hat{\alpha}_{[j,n]} := \alpha_n + \alpha_{[j,n-2]}$. It is easy to see that $I \setminus S_I$ is the type $D_{n-1,n-1}$ abelian nilradical generated by $\alpha_{[n-2,n]}$ in the root subsystem that has $\{\alpha_{[n-2,n]}\} \cup \Pi \setminus \{\alpha_{n-1}, \alpha_n\}$ as a simple system. Hence, $\text{rk}(I \setminus S_I) = n-1$ and, by Lemma 5.5, $I \cap \text{Span}(I \setminus S_I)$ is \sim -closed.

It remains to prove that all $\beta \in S_I$, $\text{rk}(\beta^\preceq) = n$ are detachable in (β^\preceq) . If $\beta = \alpha_n$ or $\beta = \alpha_n + \alpha_{n-2}$, then β is detachable in (β^\preceq) by Lemma 5.8. Otherwise, let $\beta = \hat{\alpha}_{[j,n]}$, $j \in \{1, \dots, n-3\}$. In this case we have a bipartition $(\beta^\preceq) = J_i \cup J_f$ with $J_f = (\beta^\preceq)$ and $J_i = (\beta^\preceq) \setminus (\theta_k^\preceq)$. Indeed, we have: $(\beta^\preceq) = \{\gamma \in I \mid c_{\alpha_j}(\gamma) \geq 1 \text{ or } c_{\alpha_{n-1}}(\gamma) \geq 1\}$, $J_f \setminus J_i = \{\gamma \in I \mid c_{\alpha_j}(\gamma) = 2\}$, and $J_i \setminus J_f = \{\gamma \in I \mid c_{\alpha_j}(\gamma) = 0 \text{ and } c_{\alpha_{n-1}}(\gamma) = 1\}$. Hence, if we set $H = (\check{\omega}_n - \check{\omega}_j)^\perp$, H strictly separates $J_i \setminus J_f$ from $J_f \setminus J_i$, and $J_i \cap J_f = I \cap H$. Moreover, for any $\gamma_1 \in J_i \setminus J_f$ and $\gamma_2 \in J_f \setminus J_i$ we have $c_{\alpha_{n-1}}(\gamma_2 - \gamma_1) = 0$ and $c_{\alpha_j}(\gamma_2 - \gamma_1) = 0$, hence $\gamma_2 - \gamma_1 \notin \Phi$ and $\gamma_1 \lesssim \gamma_2$ by Proposition 4.8. By Lemma 5.8, β is detachable in J_f and there exists a detaching hyperplane H^f for β in J_f such that $H^f \cup (\beta^\preceq)$ is contained in $\text{Red}(\beta)$. The proof that β is also detachable in J_i will be very similar to the proof of cases A_n . We take $H^i = (\check{\omega}_n - \check{\omega}_{n-1} - \check{\omega}_{j-1})^\perp$. Then, for each $\gamma \in (\beta^\preceq)$, if $\gamma \notin H^i$, we

have $\gamma > \beta$ and $(\gamma, \beta) = 0$. If $\gamma \in H^i$, either γ is incomparable with β , or $\gamma - \beta \in \Phi$. Hence, by Proposition 4.8, $\gamma \sim \beta$ if and only if $\gamma \notin H^i$. The simple system for $(\Phi \cap H^i)^+$ is $\{\alpha_{[n-1,n]}, \hat{\alpha}_{[j-1,n]}\} \cup \Pi \setminus \{\alpha_{j-1}, \alpha_{n-1}, \alpha_n\}$, for $j > 1$ and $\{\alpha_{[n-1,n]}\} \cup \Pi \setminus \{\alpha_{n-1}, \alpha_n\}$ for $j = 1$. For $j > 1$, the maximal roots are $\hat{\alpha}_{[1,n]}$ and $\alpha_{[j,n]} + \alpha_{[j+1,n-2]}$, while, for $j = 1$, $\Phi \cap H$ is irreducible. Hence $I \cap H^i$ is \sim -closed by Lemma 5.5.

E6. $I \cong E_{6,6}$. We choose $(S_I, \preccurlyeq) = (\alpha_6, \theta, \alpha_{\{5,6\}}, \theta - \alpha_2, \alpha_{\{4,5,6\}}, \theta - \alpha_{\{2,4\}}, \alpha_{\{2,4,5,6\}}, \theta - \alpha_{\{2,4,5\}})$. Then $I \setminus S_I$ is the type $A_{5,2}$ abelian nilradical generated by $\alpha_{[3,6]}$ in the root subsystem with simple system $\{\alpha_{[3,6]}\} \cup \Pi \setminus \{\alpha_3, \alpha_6\}$. Hence $I \cap \text{Span}(I \setminus S_I)$ is \sim -closed by Lemma 5.5. The first six β in (S_I, \preccurlyeq) are detachable in their (β^\preccurlyeq) by Lemma 5.8. Hence we have to consider only the last two roots. These are symmetric with respect to the order involution of I , hence it suffices to consider $\beta = \alpha_{\{2,4,5,6\}}$. By Proposition 4.8, $\beta \approx \gamma$ for all $\gamma \in (\beta^\preccurlyeq)$ except $\gamma = \theta - \alpha_{\{2,4,5\}}$. Then, the hyperplane $H = (\check{\omega}_6 - \check{\omega}_3)^\perp$ strictly separates β from $\theta - \alpha_{\{2,4,5\}}$ and contains all other roots in (β^\preccurlyeq) . The simple system of $(\Phi \cap H)^+$ is $\{\alpha_2, \dots, \alpha_5\} \cup \{\alpha_{[3,6]}\}$, and $(\Phi \cap H)$ is irreducible. Hence, we may apply Lemma 5.5 and conclude that β is detachable in (β^\preccurlyeq) .

E7. $I \cong E_{7,7}$. We choose $(S_I, \preccurlyeq) = (\alpha_7, \theta, \alpha_{\{6,7\}}, \theta - \alpha_1, \alpha_{\{5,6,7\}}, \theta - \alpha_{\{1,3\}}, \alpha_{\{4,5,6,7\}}, \theta - \alpha_{\{1,3,4\}}, \alpha_{\{2,4,5,6,7\}}, \theta - \alpha_{\{1,3,4,2\}}, \alpha_{\{3,4,5,6,7\}}, \theta - \alpha_{\{1,3,4,5\}}, \alpha_{\{1,3,4,5,6,7\}}, \theta - \alpha_{\{1,3,4,5,6\}})$. We note that (S_I, \preccurlyeq) consists of all β in I such that $c_{\alpha_i}(\beta) \leq 1$ for $i = 1, \dots, 7$, together with their symmetric roots, with respect to the order involution of I .

If $\beta = \alpha_{\{2,4,5,6,7\}}$ and $\beta' = \theta - \alpha_{\{1,3,4,2\}}$ (as in Figure 7), then $I \cap \text{Span}(I \setminus S_I) = (I \setminus S_I) \cup \{\beta, \beta'\}$. This is the type $D_{6,6}$ abelian nilradical generated by β in the subsystem that has $\{\beta\} \cup \Pi \setminus \{\alpha_2, \alpha_7\}$. Hence $(I \setminus S_I) \cup \{\beta, \beta'\}$ is \sim -closed by Lemma 5.5.

All roots in S_I , except $\alpha_{\{2,4,5,6,7\}}, \alpha_{\{1,3,4,5,6,7\}}$, and their symmetric roots with respect to the order involution, are detachable in their \preccurlyeq -upper cone by Lemma 5.8.

Let $\beta = \alpha_{\{2,4,5,6,7\}}$. Then (β^\preccurlyeq) has the bipartition $J_i \cup J_f$ with $J_f = (\beta^\preccurlyeq) \cap (\beta^\leq) = \{\gamma \in (\beta^\preccurlyeq) \mid c_{\alpha_2}(\gamma) \geq 1\}$ and $J_i = \{\gamma \in (\beta^\preccurlyeq) \mid c_{\alpha_2}(\gamma) \leq 1\}$. Indeed, it is easily seen that $H = (\check{\omega}_7 - \check{\omega}_2)^\perp$ contains $J_i \cap J_f$ and strictly separates $J_i \setminus J_f$ from $J_f \setminus J_i$. Moreover, it is easy to see that, for all $\gamma_1 \in J_i \setminus J_f$ and $\gamma_2 \in J_f \setminus J_i$, $\gamma_2 - \gamma_1 \in L^+(\Phi) \setminus \Phi^+$, hence $\gamma_1 \lesssim \gamma_2$. It remains to check that β is detachable in J_i and J_f and the further conditions in Definition 5.9 (2b) hold. For J_f this follows from Lemma 5.8. For J_i , the hyperplane $H^i := (\check{\omega}_7 - \check{\omega}_3)$ is a detaching hyperplane that satisfies the required conditions. Indeed, the simple system of $(\Phi \cap H^i)$ is $\{\alpha_{\{3,4,5,6,7\}}\} \cup \Pi \setminus \{\alpha_7, \alpha_3\}$, hence $I \cap H^i$ is \sim -closed by Lemma 5.5. Moreover, we can easily check that for $\gamma \in J_i \cap H^i$, either γ and β are incomparable, or $\gamma \setminus \beta \in \Phi^+$, while for all $\gamma \in J_i \setminus H^i$, we have $\gamma > \beta$ and $\gamma - \beta \notin \Phi$. Hence, we may conclude that by Lemma 4.8.

Let $\beta = \alpha_{\{1,3,4,5,6,7\}}$. In this case the hyperplane β is detachable in (β^\preccurlyeq) , since $H := (\check{\omega}_7 - \check{\omega}_2)^\perp$ satisfies the conditions of Definition 5.6. The details are similar to the previous case.

Finally, for $\beta = \theta - \alpha_{[1,4]}$ and $\theta - \alpha_{\{1,2,4,5,6\}}$ we have similar results by symmetry. \square

Figure 1. $I \cong A_{9,6}$. Here $\beta = \alpha_{[6,7]}$. The gray boxes cover the roots in $H := (\check{\omega}_6 - \check{\omega}_5 - \check{\omega}_8)^\perp$.

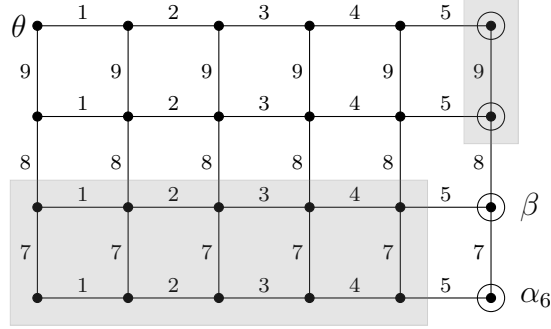


Figure 2. $I \cong C_{7,7}$. Here $\beta = \alpha_{[4,7]}$. The gray boxes cover the roots in $H := (2\check{\omega}_7 - \check{\omega}_3 - \check{\omega}_6)^\perp$.

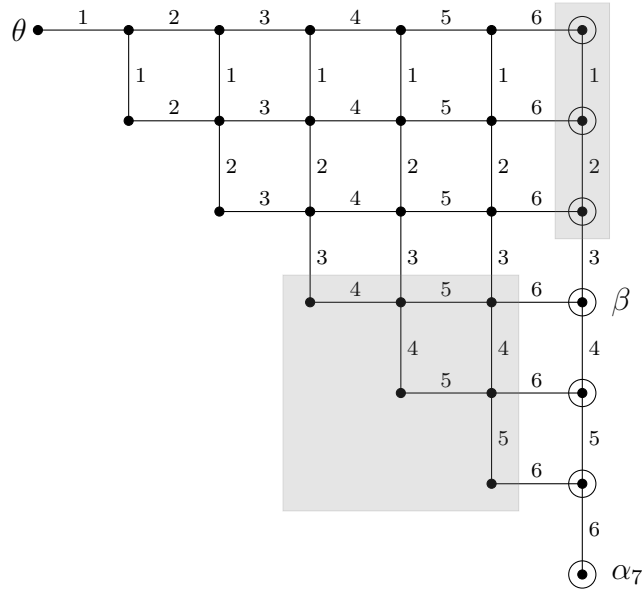


Figure 3. $I \cong B_{6,1}$ and $I \cong D_{6,1}$. In both cases $S_I = \{\alpha_1, \theta\}$. The gray boxes cover the roots in $H = (\check{\omega}_1 - \check{\omega}_2)^\perp$, for either $\beta = \alpha_1$ and $\beta = \theta$.

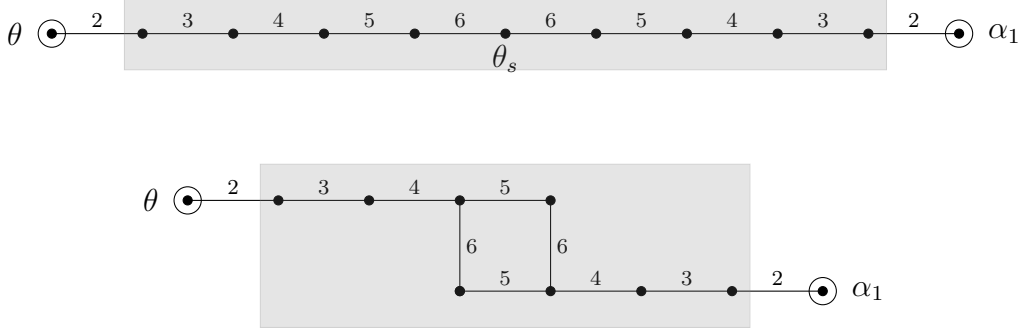


Figure 4. $I \cong D_{8,8}$. The figure represents the Hasse diagram of the whole I . The gray boxes illustrate the bipartition of (β^{\lessdot}) described in the proof, for $\beta = \hat{\alpha}_{4,8}$. The next figure represents (β^{\lessdot}) .

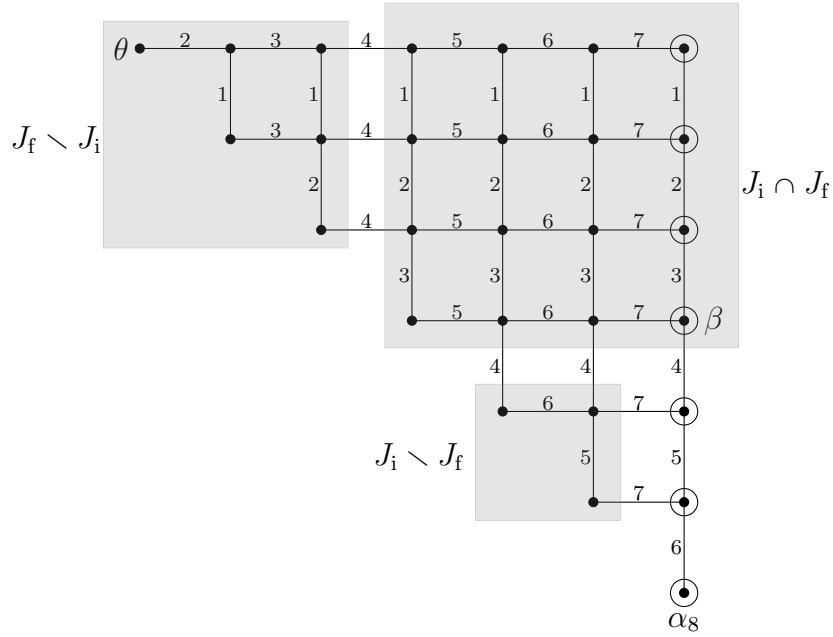


Figure 5. $I \cong D_{8,8}$. The diagram represents (β^{\preccurlyeq}) for $\beta = \hat{\alpha}_{4,8}$. The big rectangle contains the roots in J_1 and the gray parts cover the roots in $H^i = (\check{\omega}_8 - \check{\omega}_7 - \check{\omega}_3)^\perp$.

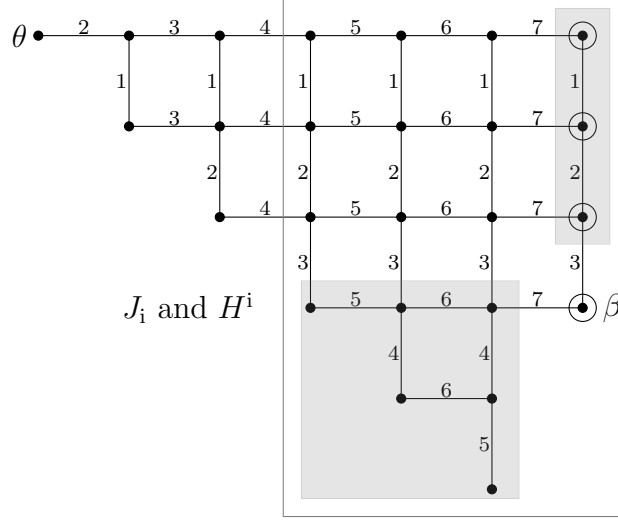


Figure 6. $I \cong E_{6,6}$. The gray rectangle covers the roots in $H = H_{\beta'} = (\check{\omega}_6 - \check{\omega}_3)^\perp$ for $\beta = \alpha_{\{2,4,5,6\}}$ and $\beta' = \theta - \alpha_{\{2,4,5\}}$. By definition (β^{\preccurlyeq}) consists of these roots plus β and β' , while (β'^{\preccurlyeq}) consists of the roots in gray rectangle plus β' .

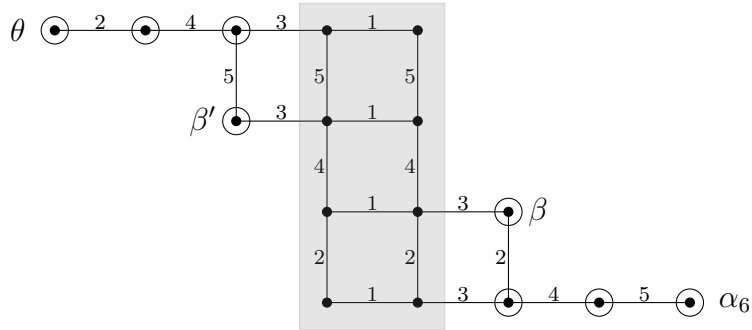


Figure 7. $I \cong E_{7,7}$. The figure represents the full Hasse diagram of I . The gray rectangles illustrate the bipartition of (β^{\lessgtr}) for $\beta = \alpha_{\{2,4,5,6,7,\}}$. The bipartition of (β'^{\lessgtr}) , for the symmetric root $\beta' = \theta - \alpha_{[1,4]}$, is similar.

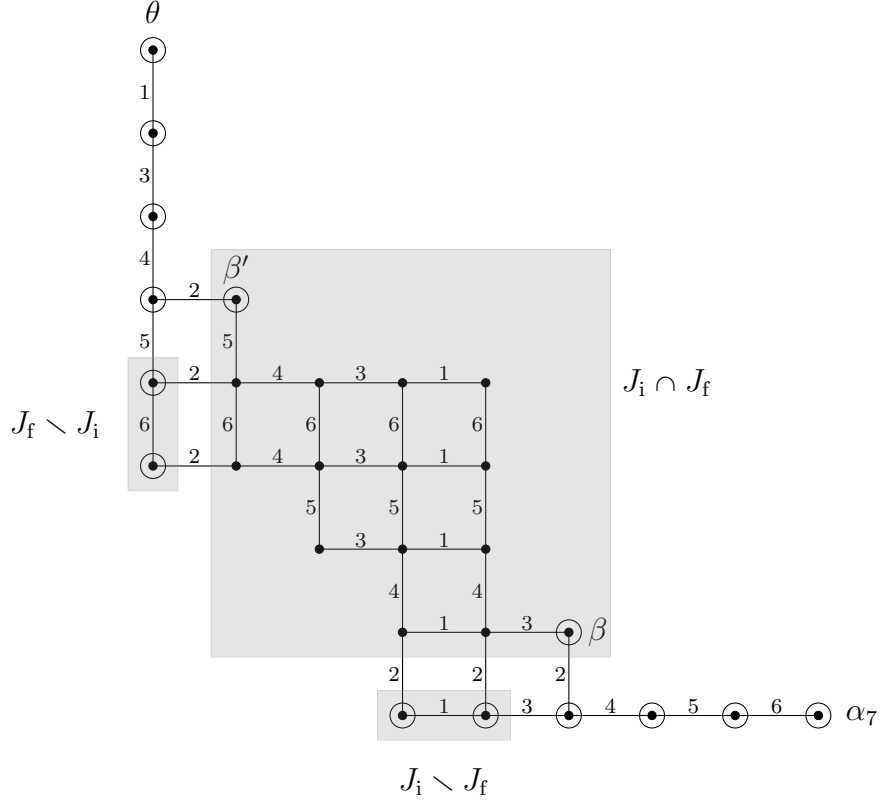


Figure 8. $I \cong E_{7,7}$. The diagram represents (β^{\lessgtr}) for $\beta = \alpha_{\{2,4,5,6,7,\}}$. The big rectangle contains the roots in J_i and gray part covers the roots in $H^i = (\tilde{\omega}_7 - \tilde{\omega}_3)^\perp$.

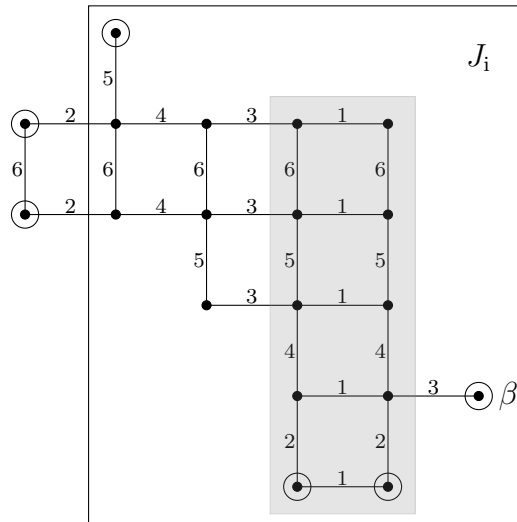
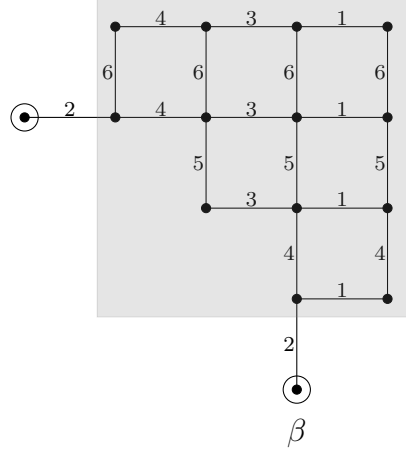


Figure 9. $I \cong E_{7,7}$. The diagram represents (β^{\lessgtr}) for $\beta = \alpha_{\{1,3,4,5,6,7,\}}$. The gray square covers the roots in $H = (\check{\omega}_7 - \check{\omega}_2)^\perp$.



6. TRIANGULATIONS OF STANDARD PARABOLIC FACETS

In this section we prove Theorems 1.1 and 1.2.

Let I be a face ideal of Φ^+ and $F_I = \text{Conv}(I)$ the corresponding standard parabolic face. For all $J \subseteq I$, let

$$\mathcal{R}_J = \{R \subseteq J \mid R \text{ reduced}\}.$$

Then, let

$$\mathcal{T}_I = \{\text{Conv}(R) \mid R \in \mathcal{R}_I, R \text{ maximal in } \mathcal{R}_I\}.$$

We will prove that \mathcal{T}_I is a triangulation of F_I .

By Propositions 3.1 and 3.2, it suffices to prove the claim when I is an abelian nilradical of Φ^+ . Henceforward, we make this assumption.

The proof is by induction on $\text{rk}(\Phi)$ and is based on the existence of triangulation orders for all facet ideals. We start with two key lemmas.

For each $J \subseteq I$ let $\text{Cone}(J)$ be the positive cone generated by J , i.e. the set of linear combinations of elements in J with nonnegative real coefficients. Moreover, let $[J]$ be the *saturation* of J , i.e.

$$[J] = \{x \in I \mid \exists y, z \in J \ y \leq x \leq z\}.$$

Lemma 6.1. *Let J be a saturated subset of I , and $\{J_i, J_f\}$ be a bipartition of J . Then $\text{Cone}(J) = \text{Cone}(J_i) \cup \text{Cone}(J_f)$.*

Proof. The claim is obvious if the bipartition is not proper, in particular if $|J| \leq 2$. The inclusion $\text{Cone}(J_i) \cup \text{Cone}(J_f) \subseteq \text{Cone}(J)$ is obvious in all cases. We prove the reverse inclusion by induction on $|J|$. It is immediate that, for any $K \subseteq J$, $\{K \cap J_i, K \cap J_f\}$ is a bipartition of K . Therefore, it suffices to prove that if the bipartition $\{J_i, J_f\}$ of J is proper, there exists a proper saturated subset K of J such that $x \in \text{Cone}(K)$.

So, let $J_i, J_f \subsetneq J$, $x \in \text{Cone}(J)$, and $x = \sum_{\beta \in J} c_\beta \beta$, with c_β nonnegative real coefficients, be a fixed expression of x . If $\{\beta \in J \mid c_\beta > 0\}$ is included in J_i or J_f we are done. Also if the saturation $[\{\beta \in J \mid c_\beta > 0\}]$ is properly included in J we are done. Hence, we assume $J = [\{\beta \in J \mid c_\beta > 0\}]$. This means that, for all $\beta \in \text{Min } J \cup \text{Max } J$, $c_\beta > 0$. Since $J_i \setminus J_f$ and $J_f \setminus J_i$ are an initial and a final section of J , we have $\text{Min}(J_i \setminus J_f) \subseteq \text{Min } J$ and $\text{Max}(J_f \setminus J_i) \subseteq \text{Max } J$. We fix $\beta_1 \in \text{Min}(J_i \setminus J_f)$ and $\beta_2 \in \text{Max}(J_f \setminus J_i)$. By Definition 5.1, and since J is saturated, there exist $\gamma_1, \gamma_2 \in J$ such that $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$ and $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$. Hence, $c_{\beta_1}\beta_1 + c_{\beta_2}\beta_2 = (c_{\beta_1} - c_{\beta_2})\beta_1 + c_{\beta_2}(\gamma_1 + \gamma_2) = (c_{\beta_2} - c_{\beta_1})\beta_2 + c_{\beta_1}(\gamma_1 + \gamma_2)$. We obtain that, if $c_{\beta_1} \geq c_{\beta_2}$, then $x \in \text{Cone}(J \setminus \{\beta_2\})$, while, if $c_{\beta_2} \geq c_{\beta_1}$, then $x \in \text{Cone}(J \setminus \{\beta_1\})$. Since β_1 and β_2 are extremal elements in J , $J \setminus \{\beta_1\}$ and $J \setminus \{\beta_2\}$ are saturated, hence the claim is proved. \square

Lemma 6.2. *Let J be a saturated subset of I , $\beta \in J$, β detachable in J , and $J_\beta = \{\beta\} \cup (\text{Red}(\beta) \cap J)$. Then $\text{Cone}(J) = \text{Cone}(J_\beta) \cup \text{Cone}(J \setminus \{\beta\})$.*

Proof. By Remark 5.7, the claim is a direct consequence of Proposition 6.1. \square

Proposition 6.3. *For all $J \subseteq I$, if J is saturated, then*

$$\text{Cone}(J) = \bigcup \{ \text{Cone}(R) \mid R \subseteq J, R \text{ reduced} \}.$$

Proof. The claim is obvious if $\text{rk}(\Phi) = 1$. We assume $\text{rk}(\Phi) \geq 2$ and the claim true for any abelian nilradical in any irreducible root system of rank strictly lower than $\text{rk}(\Phi)$.

Let $J \subseteq I$ be saturated. The inclusion “ \supseteq ” is clear, so it suffices to prove the reverse one. Let $x \in \text{Cone}(J)$, \preceq be a triangulation order on I , $\beta_0 = \max_{\preceq} \{ \beta \in J \mid x \in \text{Cone}(J \cap (\beta^\preceq)) \}$, and $J_0 = J \cap (\beta_0^\preceq)$. Then, $x \in \text{Cone}(J_0)$ and J_0 is saturated, being the intersection of two saturated sets, hence it suffice to prove the claim for J_0 . We rename $J := J_0$, so that $\beta_0 = \min_{\preceq} J$, and $x \notin \text{Cone}(J \setminus \{ \beta_0 \})$.

(a) First, we consider the case $\beta_0 \in I \setminus S_{I, \preceq}$. Let $\Psi = \Phi \cap \text{Span}(I \setminus S_{I, \preceq})$, Ψ_1, \dots, Ψ_k be the irreducible components of Ψ , $I_i = I \cap \Psi_i$, $J_i = J \cap \Psi_i$. Let $\{c_\beta \mid \beta \in J_0\}$ be a fixed set of nonnegative real coefficients such that $x = \sum_{\beta \in J_0} c_\beta \beta$ and let $x_i = \sum_{\beta \in J_i} c_\beta \beta$, for $i = 1, \dots, k$. Then, I_i is an abelian nilradical of Ψ_i^+ and J_i is saturated in it, hence, by the induction assumption, there exists a subset R_i of J_i , reduced relatively to Ψ_i , such that $x_i \in \text{Cone}(R_i)$, for $i = 1, \dots, k$. By Definition 5.9, $I \cap \Psi$ is \sim -closed and hence, by Lemma 5.4, $R_1 \cup \dots \cup R_k$ is reduced in Φ . Clearly, $x \in \text{Cone}(R_1 \cup \dots \cup R_k)$, hence we are done.

(b) Then, we assume $\text{rk}(\beta_0^\preceq) = n$. By definition, either β_0 is detachable in (β_0^\preceq) , or (β_0^\preceq) has a bipartition $\{B_i, B_f\}$ such that β_0 is a detachable element in both of B_i and B_f . In this case, $\{J \cap B_i, J \cap B_f\}$ is a bipartition of J and, by Lemma 6.1, we may fix a $B \in \{B_i, B_f\}$ such that $x \in \text{Cone}(J \cap B)$. If β_0 is detachable in (β_0^\preceq) , we set $B = (\beta_0^\preceq)$. In any case, we define $J' = J \cap B$. Then, we still have $\beta_0 = \min_{\preceq} J'$, $x \in \text{Cone}(J')$ and, in any expression of x as a linear combination of elements of J' , the coefficient of β_0 is strictly positive.

Since β_0 is detachable in B , there exists a detaching hyperplane H for β_0 in B , and it is clear that such an H is a detaching hyperplane also for β_0 in J' . By Remark 5.7, the two subsets $\{\beta_0\} \cup (J' \cap H)$ and $J' \setminus \{\beta_0\}$ form a bipartition of J' and, by Lemma 6.1, we obtain $x \in \text{Cone}(\{\beta_0\} \cup (J' \cap H))$. Hence, there exists a positive real coefficient c_0 such that $x - c_0 \beta_0 \in \text{Cone}(J' \cap H)$. Now, $J' \cap H$ is contained in the abelian nilradical $I \cap H$ of $(\Phi \cap H)^+$. Let Ψ_1, \dots, Ψ_k be the irreducible components of $\Phi \cap H$. Arguing as in case (a), we find R_1, \dots, R_k such that $R_i \subseteq J' \cap \Psi_i$, R_i is reduced relatively to Ψ_i , and $x - c_0 \beta_0 \in \text{Cone}(R_1 \cup \dots \cup R_k)$. As before, $R_1 \cup \dots \cup R_k$ is reduced in Φ , since $I \cap H$ is \sim -closed. Moreover, by definition of β_0 , $R_1 \cup \dots \cup R_k \subseteq (\beta_0^\preceq)$, hence in $(\beta_0^\preceq) \cap H$. By Definition 5.9, this subset is contained in $\text{Red}(\beta_0)$, hence $\{\beta_0\} \cup R_1 \cup \dots \cup R_k$ is reduced. This proves the claim. \square

Remark 6.4. We observe that for each $J \subseteq I$, $\text{Cone}(J) \cap F_I = \text{Conv}(J)$. Indeed, if $x = \sum_{\beta \in J} c_\beta \beta$, and α_I is the simple root of Φ^+ such that $I = (\alpha_I^\leq)$, then $\sum_{\beta \in J} c_\beta = (x, \check{\omega}_{\alpha_I})$, which is 1 for all x in F_I .

Corollary 6.5. *Let I be a facet ideal of Φ^+ and*

$$\mathcal{T}'_I = \{\text{Conv}(R) \mid R \in \mathcal{R}_I, \text{rk}(R) = n\}.$$

Then \mathcal{T}'_I is a covering of F_I .

Proof. By Proposition 6.3 and Remark 6.4, the set of all $\text{Conv}(R)$, with $R \subseteq I$ and R reduced, is a covering of F_I . By standard topological arguments, we obtain that also \mathcal{T}'_I is a covering of F_I . \square

Our next step is to prove that the set \mathcal{T}'_I defined in Corollary 6.5 is a triangulation of the standard parabolic facet F_I . For this, it remains to prove that each $T \in \mathcal{T}'_I$ is a simplex, and that the intersection of any two $T_1, T_2 \in \mathcal{T}'_I$ is a common face of T_1 and T_2 . This is proved in next two propositions.

Proposition 6.6. *Let I be an abelian nilradical of Φ^+ and R be a reduced subset of I . Then R is linearly independent.*

Proof. We prove the claim by induction on $\text{rk}(\Phi)$. The case $\text{rk}(\Phi) = 1$ is obvious. We assume $\text{rk}(\Phi) > 1$ and the claim true for irreducible root systems of rank lesser than $\text{rk}(\Phi)$. Let \preceq be a triangulation order on I and $\beta = \min_{\preceq} R$.

First we consider the case $\text{rk}(\beta^\preceq) = n$ and β detachable in (β^\preceq) . Let H be a detaching hyperplane for β in (β^\preceq) , Ψ_1, \dots, Ψ_k be the irreducible components of $\Phi \cap H$, and $R_i = (R \setminus \{\beta\}) \cap \Psi_i$, for $i = 1, \dots, k$. Then R_i is contained in the abelian nilradical $I \cap \Psi_i$ of Ψ_i^+ and is reduced, relatively to Ψ_i . By the induction assumption, it is linearly independent. Since $R \setminus \{\beta\} = R_1 \cup \dots \cup R_k$, we obtain that $R \setminus \{\beta\}$, and hence R are linearly independent.

If $\text{rk}(\beta^\preceq) = n$ and β is not detachable in (β^\preceq) , there exists a bipartition $\{J_i, J_f\}$ of (β^\preceq) such that β is a detachable element both in J_i , and in J_f . By Definition 5.1, either $R \subseteq J_i$, or $R \subseteq J_f$, hence we can argue as in the previous case.

If $\text{rk}(\beta^\preceq) < n$, then R is contained in the abelian nilradical $I \cap \text{Span}(I \setminus S_{I, \preceq})$, in $\Phi \cap \text{Span}(I \setminus S_{I, \preceq})$ and we may argue by induction as above. \square

Proposition 6.7. *Let I be an abelian nilradical of Φ^+ and R_1, R_2 be reduced subsets in I . Then, $\text{Conv}(R_1) \cap \text{Conv}(R_2) = \text{Conv}(R_1 \cap R_2)$. In particular, $\text{Conv}(R_1) \cap \text{Conv}(R_2)$ is a common face of $\text{Conv}(R_1)$ and $\text{Conv}(R_2)$.*

Proof. By Proposition 6.6, $\text{Conv}(R_i)$ is the simplex with set of vertexes R_i , for $i = 1, 2$, hence $\text{Conv}(R_1 \cap R_2)$ is common face of $\text{Conv}(R_1)$ and $\text{Conv}(R_2)$. Hence it suffices to

prove the first statement. The inclusion $\text{Conv}(R_1) \cap \text{Conv}(R_2) \supseteq \text{Conv}(R_1 \cap R_2)$ is clear. We prove the reverse one, by induction on $\text{rk}(\Phi)$.

If $\text{Cone}(R_1) \cap \text{Cone}(R_2) \subseteq \text{Cone}(R_1 \cap R_2)$ then, by Remark 6.4, the analogous relation for the convex hulls hold. So we work with cones.

For $\text{rk}(\Phi) = 1$ the claim is obvious. Let $\text{rk}(\Phi) > 1$, \preceq be a fixed triangulation order on I , and $\beta = \min_{\preceq}(R_1 \cup R_2)$.

(a) If $\text{rk}(\beta^\preceq) < n$, then $R_1, R_2 \subseteq I \setminus S_{I, \preceq}$. Let Ψ_1, \dots, Ψ_k be the connected components of $\Phi \cap \text{Span}(I \setminus S_{I, \preceq})$. Let $R_{j,i} = R_j \cap \Psi_i$, for $j = 1, 2$ and $i = 1, \dots, k$. Each $R_{j,i}$ is a reduced subset in the abelian nilradical $I \cap \Psi_i$ of Ψ_i^+ , hence by the induction assumption $\text{Cone}(R_{1,i}) \cap \text{Cone}(R_{2,i}) \subseteq \text{Cone}(R_{1,i} \cap R_{2,i})$, for each i in $\{1, \dots, k\}$. This implies directly the inclusion $\text{Cone}(R_1) \cap \text{Cone}(R_2) \subseteq \text{Cone}(R_1 \cap R_2)$.

(b) Next, let $\text{rk}(\beta^\preceq) = n$, β be detachable in (β^\preceq) , H be a detaching hyperplane, and $\overline{R}_i = R_i \cap H$ for $i = 1, 2$. Then it is clear that R_i is contained in one of the two closed half spaces determined by H in E , hence it is easily seen that $\text{Cone}(R_i) \cap H = \text{Cone}(\overline{R}_i)$. Moreover, if $\beta \in R_i$, then $R_i \setminus \{\beta\} = \overline{R}_i$. Now we distinguish two possibilities.

(b1) If $\beta = \min_{\preceq} R_i < \min_{\preceq} R_{i'}$, with $\{i, i'\} = \{1, 2\}$, then R_1 and R_2 are weakly separated by H . Hence, $R_1 \cap R_2 = \overline{R}_1 \cap \overline{R}_2$ and, moreover, $\text{Cone}(R_1) \cap \text{Cone}(R_2) = \text{Cone}(R_1) \cap H \cap \text{Cone}(R_2) = \text{Cone}(\overline{R}_1) \cap \text{Cone}(\overline{R}_2)$. Arguing as in case (a), with $\Phi \cap H$ in place of $\Phi \cap \text{Span}(I \setminus S_{I, \preceq})$ and \overline{R}_i in place of R_i , by the induction assumption we obtain $\text{Cone}(\overline{R}_1) \cap \text{Cone}(\overline{R}_2) \subseteq \text{Cone}(\overline{R}_1 \cap \overline{R}_2)$, and hence the claim.

(b2) If $\beta = \min_{\preceq} R_1 = \min_{\preceq} R_2$, then for all $x \in \text{Cone}(R_1) \cap \text{Cone}(R_2)$ there exist $c_i \in \mathbb{R}$ and $\overline{x}_i \in \text{Cone}(\overline{R}_i)$ ($i = 1, 2$) such that $x = c_1\beta + \overline{x}_1 = c_2\beta + \overline{x}_2$. Since $\overline{x}_1, \overline{x}_2 \in H$ and $\beta \notin H$, we must have $c_1 = c_2$ and hence $\overline{x}_1 = \overline{x}_2$. It follows $\overline{x}_1 \in \text{Cone}(\overline{R}_1 \cap \overline{R}_2)$ and hence $x \in \text{Cone}(R_1 \cap R_2)$.

(c) Finally, let $\text{rk}(\beta^\preceq) = n$, β not be detachable in (β^\preceq) , and $\{J_i, J_f\}$ be a bipartition of (β^\preceq) . By definition, each of R_1 and R_2 is contained in exactly one of J_i and J_f . If both are contained in J_i , or both in J_f , we are reduced to case (b). Otherwise, we may assume $R_1 \subseteq J_i$, $R_2 \subseteq J_f$, $R_1 \cap (J_i \setminus J_f) \neq \emptyset$, and $R_2 \cap (J_f \setminus J_i) \neq \emptyset$. Let H be a separating hyperplane for the bipartition $\{J_i, J_f\}$, and $\overline{R}_i = R_i \cap H$, for $i = 1, 2$. For a fixed i in $\{1, 2\}$, $\text{Conv}(R_i)$ is contained in one of the half-spaces determined by H in E , hence $H \cap \text{Cone}(R_i) = \text{Cone}(\overline{R}_i)$. Moreover, $\text{Cone}(R_1)$ and $\text{Cone}(R_2)$ belong to opposite half-spaces with respect to H , hence $R_1 \cap R_2 = \overline{R}_1 \cap \overline{R}_2$ and $\text{Cone}(R_1) \cap \text{Cone}(R_2) = \text{Cone}(\overline{R}_1) \cap \text{Cone}(\overline{R}_2)$. Arguing by induction as in case (b1), we obtain $\text{Cone}(\overline{R}_1) \cap \text{Cone}(\overline{R}_2) \subseteq \text{Cone}(\overline{R}_1 \cap \overline{R}_2)$, and hence the claim. \square

Corollary 6.3 and Propositions 6.6 and 6.7 imply directly the following theorem, which is Theorem 1.1.

Theorem 6.8. *Let I be a facet ideal in Φ and*

$$\mathcal{T}'_I = \{\text{Conv}(R) \mid R \in \mathcal{R}_I, \text{rk}(R) = n\}.$$

Then \mathcal{T}'_I is a triangulation of the facet ideal F_I .

Corollary 6.9. *Each reduced subset in I is contained in a maximal reduced subset. Moreover, each maximal reduced subset in I has rank n , in particular is a linear basis of E .*

Proof. Let R_0 be a reduced subset in I such that $\text{rk}(R_0) < n$. Let $x = \sum_{\beta \in R_0} c_\beta \beta$ with $c_\beta > 0$ for all $\beta \in R_0$. By Corollary 6.5, there exists a reduced subset R in I such that $\text{rk}(R) = n$ and $x \in \text{Conv}(R)$. Then, by Proposition 6.7, $x \in \text{Conv}(R_0 \cap R)$. It follows that $R_0 \cap R = R_0$, hence R_0 is not maximal. The rest of the claim follows from Proposition 6.6. \square

We can finally prove the following result, which is clearly equivalent to Theorem 1.2. The proof refers to the case by case analysis of Proposition 5.12.

Theorem 6.10. *Let I be a facet ideal in Φ and R be a maximal reduced subset in I . Then R is a \mathbb{Z} -basis of the sub-lattice of $L(\Phi)$ generated by $(\Pi \setminus \{\alpha_I\}) \cup \{m_{\alpha_I} \alpha_I\}$, where α_I is the simple root such that $I = V_{\alpha_I}$.*

Proof. By Proposition 3.2 and Remark 3.3, it suffices to prove the claim in case I is an abelian nilradical of Φ^+ , i.e. $m_{\alpha_I} = 1$. Under this assumption, we have to prove that R is a \mathbb{Z} -basis of $L(\Phi)$.

Let \preceq be a triangulation order of I and $\beta = \min_{\preceq} R$. If β is detachable in (β^\preceq) , let $J = (\beta^\preceq) \setminus \{\beta\}$. If β is not detachable in (β^\preceq) , let $\{J_i, J_f\}$ be a bipartition of (β^\preceq) such that β belongs to J_i and J_f and is detachable in them. In this case, R is contained in exactly one of J_i and J_f : we define $J = J_i$ if $R \subseteq J_i$, and $J = J_f$ otherwise. In any case, there exists a hyperplane H such that $\text{Red}(\beta) \cap J = H \cap J$, hence $R \setminus \{\beta\}$ is a reduced subset in the abelian nilradical $I \cap H$ of $(\Phi \cap H)^+$. Since $\text{rk}(R \setminus \{\beta\}) = n - 1$, also $\text{rk}(I \cap H) = \text{rk}(\Phi \cap H) = n - 1$. In particular $I \cap H$ has nontrivial intersection with each irreducible component of $\Phi \cap H$. By Lemma 3.4, each of these intersections is a nontrivial abelian nilradical in its irreducible component, hence, by induction on the dimension, $R \setminus \{\beta\}$ is a \mathbb{Z} -basis of $L(\Phi \cap H)$.

Now, we first consider the case in which β is long and is equal to $\min J$ or $\max J$ with respect to standard partial order. In this case, as seen in the proof of Lemma 5.8, $H = (\beta^\vee - \tilde{\omega}_{\alpha_I})^\perp$. It follows directly that all simple roots different from α_I and perpendicular to β belong to H . For all other simple roots $\alpha \neq \alpha_I$, either $(\alpha, \beta^\vee) = 1$ and $\beta - \alpha \in H$, or $(\alpha, \beta^\vee) = -1$ and $\beta + \alpha \in H$. Since the \mathbb{Z} -span of $R \setminus \{\beta\}$ contains all the roots in H , we obtain that the \mathbb{Z} -span of R contains $(\Pi \setminus \alpha_I) \cup \{\beta\}$, and hence contains Π , as claimed.

In the remaining cases, looking the proof of Proposition 5.12, we can directly check that for all $\alpha \in \Pi \setminus \{\alpha_I\}$, if $\beta + \alpha \notin \Phi$ then $\alpha \in H$ and, otherwise, $\beta + \alpha \in \Phi \cap H$. Arguing as in the previous case, we easily obtain that R is a \mathbb{Z} -basis of $L(\Phi)$. \square

7. CONCLUDING REMARKS

Via the action of the Weyl group, we may transport a triangulation of a standard parabolic facet to all facets in its orbit. Hence from the triangulations of all parabolic facets we obtain a triangulation \mathcal{T} of the whole boundary $\partial\mathcal{P}$ of the root polytope \mathcal{P} . Clearly, such a \mathcal{T} is not unique, since the way of transporting a triangulation of a standard parabolic facet to the facets in its orbits is not unique; the possible ways correspond to the systems of representatives of the left cosets of W modulo the stabilizer of the standard parabolic facet. For a fixed \mathcal{T} , for each $T \in \mathcal{T}$, let V_T be the set of vertexes of T and $T_0 = \text{Conv}(V_T \cup \{\underline{0}\})$. Then, clearly $\mathcal{T}_0 := \{T_0 \mid T \in \mathcal{T}\}$ is a triangulation of \mathcal{P} . Thus, the explicit enumeration of the maximal reduced subsets of facet ideals, together with the above Theorem 6.10 and the results in [5] would allow to compute the volume of \mathcal{P} . For the root types A and C, this is done in [6]. For the remaining types, it will be done in a next paper. In fact, the proof of Proposition 5.12 gives an explicit procedure for enumerating the reduced subsets.

In [6], with a suitable choice of the systems of representatives of the left cosets of W modulo the stabilizers of the standard parabolic facets, we have obtained a triangulation of \mathcal{P} that restricts to a triangulation of the positive root polytope \mathcal{P}^+ , which by definition is $\text{Conv}(\Phi^+ \cup \{\underline{0}\})$. In fact, this is a proof that, for the types A and C, the intersection of \mathcal{P} with the cone on Φ^+ is equal to \mathcal{P}^+ . This is one of the special properties of the root polytope that hold only for the types A and C (see also [7]). In fact, it is easy to see that, for all other root types, \mathcal{P}^+ is properly contained in $\mathcal{P} \cap \text{Cone}(\Phi^+)$ [4]. Hence, in these cases, from the standard parabolic facets, we cannot obtain any triangulation of the positive root polytope.

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